

STRUCTURE OF $\text{Gal}(\mathbb{k}_2^{(2)}/\mathbb{k})$ FOR SOME FIELDS
 $\mathbb{k} = \mathbb{Q}(\sqrt{2p_1p_2}, i)$ **WITH $Cl_2(\mathbb{k}) \simeq (2, 2, 2)$**

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ABSTRACT. Let $p_1 \equiv p_2 \equiv 5 \pmod{8}$ be different primes. Put $i = \sqrt{-1}$ and $d = 2p_1p_2$, then the bicyclic biquadratic field $\mathbb{k} = \mathbb{Q}(\sqrt{d}, i)$ has an elementary abelian 2-class group of rank 3. In this paper we determine the nilpotency class, the coclass, the generators and the structure of the non-abelian Galois group $\text{Gal}(\mathbb{k}_2^{(2)}/\mathbb{k})$ of the second Hilbert 2-class field $\mathbb{k}_2^{(2)}$ of \mathbb{k} . We study the capitulation problem of the 2-classes of \mathbb{k} in its seven unramified quadratic extensions \mathbb{K}_i and in its seven unramified bicyclic biquadratic extensions \mathbb{L}_i .

1. INTRODUCTION

Let k be an algebraic number field and $Cl_2(k)$ be its 2-class group i.e. the Sylow 2-subgroup of the ideal class group, $Cl(k)$, of k . Denote by $k_2^{(1)}$ the Hilbert 2-class field of k and by $k_2^{(2)}$ its second Hilbert 2-class field. Put $G = \text{Gal}(k_2^{(2)}/k)$ and let G' denote its derived group, then it is well known that $C/G' \simeq Cl_2(k)$. The knowledge of G , its structure and its generators solves a lot of problems in number theory as capitulation problems, the finiteness or not of the towers of number fields and the structures of the 2-class groups of the unramified extensions of k within $k_2^{(1)}$. For particular types of fields k , for example, fields with $Cl_2(k) \simeq (2, 2)$, the structure of G has been completely determined (see [13]). The success in this case is in part due to the fact that, contrary to most other cases, 2-groups whose abelianization is $(2, 2)$ are well understood, cf. [15] and [14].

If one considers another case, namely where $k = \mathbb{Q}(\sqrt{d}, i)$ and $Cl_2(k) \simeq (2, 2, 2)$, for some square-free integer d , then the situation is very different and very difficult when compared with the case described above; moreover there is no known way (to our knowledge) to determine the structure of G . Our aim in the present paper is to determine the isomorphism types of the second 2-class group of certain number

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fields $k = \mathbb{Q}(\sqrt{d}, i)$, to give the structure and generators of G and we will explicitly determine $\ker j_{K/k}$, the kernel of the natural class extension homomorphism $j_{K/k} : Cl(k) \rightarrow Cl(K)$, where K is an unramified extension of k within $k_2^{(1)}$. It should be noted that the determination of $\ker j_{K/k}$ is not always easy to do, especially when $K = k(\sqrt{a + b\sqrt{-1}})$ or $K = k(\sqrt{a + b\sqrt{-1}}, \sqrt{a' + b'\sqrt{-1}})$, with some positive integers a, b, a' and b' .

Let m be a square-free integer and K be a number field. Throughout this paper, we adopt the following notations:

- $h(m)$, (resp. $h(K)$): the 2-class number of $\mathbb{Q}(\sqrt{m})$ (resp. K).
- \mathcal{O}_K : the ring of integers of K .
- E_K : the unit group of \mathcal{O}_K .
- W_K : the group of roots of unity contained in K .
- ω_K : the order of W_K .
- K^+ : the maximal real subfield of K , if it is a CM-field.
- $Q_K = [E_K : W_K E_{K^+}]$ is Hasse's unit index, if K is a CM-field.
- $q(K/\mathbb{Q}) = [E_K : \prod_i^s E_{k_i}]$ is the unit index of K , if K is multiquadratic, where k_i are the quadratic subfields of K .
- $K^{(*)}$: the genus field of K .
- $\mathbf{Cl}_2(K)$: the 2-class group of K .
- ε_m : the fundamental unit of $\mathbb{Q}(\sqrt{m})$.
- $i = \sqrt{-1}$.
- FSU: denotes a fundamental system of units.

2. MAIN RESULTS

Let $p_1 \equiv p_2 \equiv 5 \pmod{8}$ be different primes, then there exist some positive integers e, f, g and h such that $p_1 = e^2 + 4f^2$ and $p_2 = g^2 + 4h^2$. Let $p_1 = \pi_1 \pi_2$ and $p_2 = \pi_3 \pi_4$, where $\pi_1 = e + 4if$ and $\pi_2 = e - 4if$ (resp. $\pi_3 = g + 4ih$ and $\pi_4 = g - 4ih$) are conjugate prime elements in the cyclotomic field $k = \mathbb{Q}(i)$ dividing p_1 (resp. p_2). Denote by \mathbb{k} the imaginary bicyclic biquadratic field $\mathbb{Q}(\sqrt{d}, i)$, where $d = 2p_1 p_2$, its three quadratic subfields are $k = \mathbb{Q}(i)$, $k_0 = \mathbb{Q}(\sqrt{d})$ and $\bar{k}_0 = \mathbb{Q}(\sqrt{-d})$. Let $\mathbb{k}_2^{(1)}$ be the Hilbert 2-class field of \mathbb{k} , $\mathbb{k}_2^{(2)}$ its second Hilbert 2-class field and G be the Galois group of $\mathbb{k}_2^{(2)}/\mathbb{k}$. According to [5], \mathbb{k} has an elementary abelian 2-class group $\mathbf{Cl}_2(\mathbb{k})$ of rank 3, that is, of type $(2, 2, 2)$. In an earlier paper [4] we have proved that the 2-class field tower of \mathbb{k} has length 2, the order of G is greater

than or equal to 64, we have given necessary and sufficient conditions to have G of order 64 and we have shown that if \mathbb{K} is an unramified quadratic extension of \mathbb{k} other than $\mathbb{K}_3 = \mathbb{k}(\sqrt{2})$, then $\mathbf{Cl}_2(\mathbb{K})$ is of type $(2, 4)$ or $(2, 2, 2)$. In this paper we complete this study by determining the structure of G , the abelian type invariants of the 2-class groups of all the unramified extensions of \mathbb{k} within $\mathbb{k}_2^{(1)}$ and the kernel of the natural class extension homomorphism $j_{\mathbb{K}/\mathbb{k}} : \mathbf{Cl}_2(\mathbb{k}) \longrightarrow \mathbf{Cl}_2(\mathbb{K})$, where \mathbb{K} is an unramified extension of \mathbb{k} within $\mathbb{k}_2^{(1)}$. The main results of this paper are Theorems 2 and 3 below; whereas Theorem 1 is proved in [2], [5] and [17].

2.1. Unramified extensions of \mathbb{k} . The first and the second assertions of the following theorem hold according to [17] and [5] respectively, the others are shown in [2].

Theorem 1. *Let p_1, p_2 be as above.*

- (1) *The 2-class groups of k_0, \bar{k}_0 are of type $(2, 2)$.*
- (2) *The 2-class group, $\mathbf{Cl}_2(\mathbb{k})$, of \mathbb{k} is of type $(2, 2, 2)$.*
- (3) *The discriminant of \mathbb{k} is: $\text{disc}(\mathbb{k}) = \text{disc}(k) \cdot \text{disc}(k_0) \cdot \text{disc}(\bar{k}_0) = 2^8 p_1^2 p_2^2$.*
- (4) *\mathbb{k} has seven unramified quadratic extensions within its Hilbert 2-class field $\mathbb{k}_2^{(1)}$. They are given by:*

$$\begin{aligned} \mathbb{K}_1 &= \mathbb{k}(\sqrt{p_1}), & \mathbb{K}_2 &= \mathbb{k}(\sqrt{p_2}), & \mathbb{K}_3 &= \mathbb{k}(\sqrt{2}), \\ \mathbb{K}_4 &= \mathbb{k}(\sqrt{\pi_1 \pi_3}), & \mathbb{K}_5 &= \mathbb{k}(\sqrt{\pi_1 \pi_4}), & \mathbb{K}_6 &= \mathbb{k}(\sqrt{\pi_2 \pi_3}) \text{ and } \mathbb{K}_7 = \mathbb{k}(\sqrt{\pi_2 \pi_4}). \end{aligned}$$

- (5) *$\mathbb{K}_1, \mathbb{K}_2, \mathbb{K}_3$ are intermediate fields between \mathbb{k} and its genus field $\mathbb{k}^{(*)}$. The fields $\mathbb{K}_4 \simeq \mathbb{K}_7$ and $\mathbb{K}_5 \simeq \mathbb{K}_6$ are pairwise conjugate and thus isomorphic. Consequently $\mathbb{K}_1, \mathbb{K}_2, \mathbb{K}_3$ are absolutely abelian, whereas $\mathbb{K}_4, \mathbb{K}_5, \mathbb{K}_6, \mathbb{K}_7$ are non-normal over \mathbb{Q} .*
- (6) *\mathbb{k} has seven unramified bicyclic biquadratic extensions within its Hilbert 2-class field $\mathbb{k}_2^{(1)}$. One of them is*

$$\mathbb{L}_1 = \mathbb{K}_1 \cdot \mathbb{K}_2 \cdot \mathbb{K}_3 = \mathbb{k}^{(*)} = \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{q}, \sqrt{-1}),$$

the absolute genus field of \mathbb{k} and the others are given by:

$$\mathbb{L}_2 = \mathbb{K}_1 \cdot \mathbb{K}_4 \cdot \mathbb{K}_6, \quad \mathbb{L}_3 = \mathbb{K}_1 \cdot \mathbb{K}_5 \cdot \mathbb{K}_7, \quad \mathbb{L}_4 = \mathbb{K}_2 \cdot \mathbb{K}_4 \cdot \mathbb{K}_5 \text{ and } \mathbb{L}_5 = \mathbb{K}_2 \cdot \mathbb{K}_6 \cdot \mathbb{K}_7$$

are non-normal over \mathbb{Q} ; moreover $\mathbb{L}_2 \simeq \mathbb{L}_3$ and $\mathbb{L}_4 \simeq \mathbb{L}_5$.

$$\mathbb{L}_6 = \mathbb{K}_3 \cdot \mathbb{K}_4 \cdot \mathbb{K}_7 \text{ and } \mathbb{L}_7 = \mathbb{K}_3 \cdot \mathbb{K}_5 \cdot \mathbb{K}_6 \text{ are absolutely Galois.}$$

2.2. Structure of $G = \text{Gal}(\mathbb{k}_2^{(2)}/\mathbb{k})$. Let \mathcal{H}_0 (resp. $\mathcal{H}_1, \mathcal{H}_2$) denote the prime ideal of \mathbb{k} lying above $1+i$ (resp. π_1, π_2). Write $q = q(\mathbb{K}_3^+/\mathbb{Q})$ for simplicity.

Theorem 2. *Keep the preceding assumptions. Then*

- (1) $\text{Cl}_2(\mathbb{k}) = \langle [\mathcal{H}_0], [\mathcal{H}_1], [\mathcal{H}_2] \rangle \simeq (2, 2, 2)$.
- (2) $\text{Cl}_2(\mathbb{K}_3)$ is of type $\begin{cases} (2^m, 2^{n+1}) & \text{if } q = 1, \\ (2^{\min(m, n+1)}, 2^{\max(m+1, n+2)}) & \text{if } q = 2, \end{cases}$
where n and m are determined by $2^{m+1} = h(-p_1 p_2)$, $m \geq 2$, and $2^n = h(p_1 p_2)$, $n \geq 1$; and either $n \geq 3$ or $m \geq 3$.
- (3) The length of the 2-class field tower of \mathbb{k} is 2.
- (4) $G = \text{Gal}(\mathbb{k}_2^{(2)}/\mathbb{k})$ is given by:

(i) If $q = 1$, then

$$G = \langle \rho, \tau, \sigma : \rho^4 = \sigma^{2^m} = \tau^{2^{n+1}} = 1, \rho^2 = \psi, [\tau, \sigma] = 1, \\ [\rho, \sigma] = \sigma^2, [\rho, \tau] = \tau^2 \rangle, \text{ where}$$

$$\psi = \begin{cases} \sigma^{2^{m-1}} & \text{if } (\frac{p_1}{p_2}) = 1 \text{ and } N(\varepsilon_{p_1 p_2}) = 1, \\ \tau^{2^n} \sigma^{2^{m-1}} & \text{if } (\frac{p_1}{p_2}) = -1 \text{ or } (\frac{p_1}{p_2}) = 1 \text{ and } N(\varepsilon_{p_1 p_2}) = -1. \end{cases}$$

(ii) If $q = 2$, then

$$G = \langle \rho, \tau, \sigma : \rho^4 = \sigma^{2^{m+1}} = \tau^{2^{n+2}} = 1, \sigma^{2^m} = \tau^{2^{n+1}}, \\ \rho^2 = \tau^{2^n} \sigma^{2^{m-1}}, [\tau, \sigma] = 1, [\rho, \sigma] = \sigma^{-2}, [\rho, \tau] = \tau^2 \rangle.$$

- (5) The derived group of G is $G' = \langle \sigma^2, \tau^2 \rangle$ and $\text{Cl}_2(\mathbb{k}_2^{(1)})$ is of type $\begin{cases} (2^{m-1}, 2^n) & \text{if } q = 1, \\ (2^{\min(m, n+1)-1}, 2^{\max(m+1, n+2)-1}) & \text{if } q = 2. \end{cases}$
- (6) The coclass of G is 3 and its nilpotency class is $\begin{cases} \max(n, m-1) + 1 & \text{if } q = 1, \\ \max(n+1, m) + 1 & \text{if } q = 2. \end{cases}$

2.3. Abelian type invariants and capitulation kernels. Let N_j denote the subgroup $N_{\mathbb{K}_j/\mathbb{k}}(\text{Cl}_2(\mathbb{K}_j))$ of $\text{Cl}_2(\mathbb{k})$ and $\kappa_{\mathbb{K}}$ denote the kernel of the natural class extension homomorphism $j_{\mathbb{K}/\mathbb{k}} : \text{Cl}_2(\mathbb{k}) \longrightarrow \text{Cl}_2(\mathbb{K})$, where \mathbb{K} is an unramified extension of \mathbb{k} within $\mathbb{k}_2^{(1)}$.

Theorem 3. *Let $2^n = h(p_1 p_2)$, $2^{m+1} = h(-p_1 p_2)$, where $n \geq 1$ and $m \geq 2$.*

- (1) $\#\kappa_{\mathbb{K}_j} = 4$, for all $j \neq 3$. If $j = 3$, then $\#\kappa_{\mathbb{K}_3} = \begin{cases} 4 & \text{if } q = 1, \\ 2 & \text{if } q = 2. \end{cases}$

- (2) All the extensions \mathbb{K}_j satisfy Taussky's condition (A) i.e. $\#\kappa_{\mathbb{K}_j} \cap N_j > 1$, for details see [16].
- (3) The order of $\kappa_{\mathbb{L}_j}$ is 8 (total 2-capitulation), for all j , and \mathbb{L}_j are of type (A).
- (4) The abelian type invariants of the 2-class groups $\mathbf{Cl}_2(\mathbb{K}_j)$ are given by:

- (i) $\mathbf{Cl}_2(\mathbb{K}_1) \simeq \mathbf{Cl}_2(\mathbb{K}_2) \simeq \begin{cases} (2, 2, 2) & \text{if } \left(\frac{p_1}{p_2}\right) = 1, \\ (2, 4) & \text{otherwise.} \end{cases}$
- (ii) If $\left(\frac{p_1}{p_2}\right) = 1$, then $\mathbf{Cl}_2(\mathbb{K}_4)$, $\mathbf{Cl}_2(\mathbb{K}_5)$, $\mathbf{Cl}_2(\mathbb{K}_6)$ and $\mathbf{Cl}_2(\mathbb{K}_7)$ are of type $(2, 2, 2)$ if $\left(\frac{\pi_1}{\pi_3}\right) = -1$, and of type $(2, 4)$ otherwise.
- (iii) Assume $\left(\frac{p_1}{p_2}\right) = -1$.

$$\text{If } \left(\frac{\pi_1}{\pi_3}\right) = -1, \text{ then } \begin{cases} \mathbf{Cl}_2(\mathbb{K}_4) \simeq \mathbf{Cl}_2(\mathbb{K}_7) \simeq (2, 4), \\ \mathbf{Cl}_2(\mathbb{K}_5) \simeq \mathbf{Cl}_2(\mathbb{K}_6) \simeq (2, 2, 2). \end{cases}$$

$$\text{If } \left(\frac{\pi_1}{\pi_3}\right) = 1, \text{ then } \begin{cases} \mathbf{Cl}_2(\mathbb{K}_4) \simeq \mathbf{Cl}_2(\mathbb{K}_7) \simeq (2, 2, 2), \\ \mathbf{Cl}_2(\mathbb{K}_5) \simeq \mathbf{Cl}_2(\mathbb{K}_6) \simeq (2, 4). \end{cases}$$

- (5) The abelian type invariants of the 2-class groups $\mathbf{Cl}_2(\mathbb{L}_j)$ are given by:

- (i) $\mathbf{Cl}_2(\mathbb{L}_1) = \mathbf{Cl}_2(\mathbb{K}^{(*)}) \simeq \begin{cases} (2^m, 2^n) & \text{if } q = 1, \\ (2^{\min(m,n)}, 2^{\max(m+1,n+1)}) & \text{if } q = 2. \end{cases}$
- (ii) If $\left(\frac{p_1}{p_2}\right) = -1$ or $\left(\frac{p_1}{p_2}\right) = \left(\frac{\pi_1}{\pi_3}\right) = 1$, then $\mathbf{Cl}_2(\mathbb{L}_2)$, $\mathbf{Cl}_2(\mathbb{L}_3)$, $\mathbf{Cl}_2(\mathbb{L}_4)$ and $\mathbf{Cl}_2(\mathbb{L}_5)$ are of type $(2, 4)$.
If $\left(\frac{p_1}{p_2}\right) = -\left(\frac{\pi_1}{\pi_3}\right) = 1$, then $\mathbf{Cl}_2(\mathbb{L}_2)$, $\mathbf{Cl}_2(\mathbb{L}_3)$, $\mathbf{Cl}_2(\mathbb{L}_4)$ and $\mathbf{Cl}_2(\mathbb{L}_5)$ are of type $(2, 2, 2)$.

- (iii) (a) Assume $q = 2$, so $\mathbf{Cl}_2(\mathbb{L}_6)$ and $\mathbf{Cl}_2(\mathbb{L}_7)$ are of type $(2, 2^{n+2})$ if $\left(\frac{p_1}{p_2}\right) = 1$, otherwise we have:

$$\mathbf{Cl}_2(\mathbb{L}_6) \simeq \begin{cases} (4, 4) & \text{if } \left(\frac{\pi_1}{\pi_3}\right) = 1, \\ (2, 8) & \text{if } \left(\frac{\pi_1}{\pi_3}\right) = -1, \end{cases} \quad \mathbf{Cl}_2(\mathbb{L}_7) \simeq \begin{cases} (2, 8) & \text{if } \left(\frac{\pi_1}{\pi_3}\right) = 1, \\ (4, 4) & \text{if } \left(\frac{\pi_1}{\pi_3}\right) = -1. \end{cases}$$

- (b) Assume $q = 1$.

$$\text{If } \left(\frac{1+i}{\pi_1}\right)\left(\frac{1+i}{\pi_3}\right) = 1, \text{ then } \begin{cases} \mathbf{Cl}_2(\mathbb{L}_6) \simeq (2^{m-1}, 2^{n+1}), \\ \mathbf{Cl}_2(\mathbb{L}_7) \simeq (2^{\min(m-1,n)}, 2^{\max(m,n+1)}). \end{cases}$$

$$\text{If } \left(\frac{1+i}{\pi_1}\right)\left(\frac{1+i}{\pi_3}\right) = -1, \text{ then } \begin{cases} \mathbf{Cl}_2(\mathbb{L}_6) \simeq (2^{\min(m-1,n)}, 2^{\max(m,n+1)}), \\ \mathbf{Cl}_2(\mathbb{L}_7) \simeq (2^{m-1}, 2^{n+1}). \end{cases}$$

3. PRELIMINARY RESULTS

Let p_1, p_2 be different primes satisfying the conditions mentioned at the beginning of §2, and set $k_1 = \mathbb{Q}(\sqrt{p_1 p_2})$, $\bar{k}_1 = \mathbb{Q}(\sqrt{-p_1 p_2})$. Put $\varepsilon_{2p_1 p_2} = x + y\sqrt{2p_1 p_2}$

and $\varepsilon_{p_1 p_2} = a + b\sqrt{p_1 p_2}$. Let $\left(\frac{g, h}{p}\right)$ denote the quadratic Hilbert symbol for the prime p .

Lemma 1. *Let ε_d denote the fundamental unit of k_0 . Then*

- (i) $N(\varepsilon_d) = -1$.
- (ii) *If $N(\varepsilon_{p_1 p_2}) = 1$, then $2p_1(a \pm 1)$ (i.e. $2p_2(a \mp 1)$) is a square in \mathbb{N} .*

Proof. (i) See [5].

(ii) As $N(\varepsilon_{p_1 p_2}) = 1$, so $\left(\frac{p_1}{p_2}\right) = 1$ and $a^2 - 1 = b^2 p_1 p_2$, so:

- (a) If $a \pm 1$ is a square in \mathbb{N} , then $\left(\frac{2}{p_1}\right) = -1$, which is false.
- (b) If $p_1(a \pm 1)$ is a square in \mathbb{N} , then $\left(\frac{p_1}{p_2}\right) = \left(\frac{2}{p_1}\right) = -1$, which is absurd. And the result derived. \square

Lemma 2. *If $\left(\frac{p_1}{p_2}\right) = 1$, then $\left(\frac{p_1}{p_2}\right)_4 \left(\frac{p_2}{p_1}\right)_4 = \left(\frac{\pi_1}{\pi_3}\right)$.*

Proof. From [17] we get $\left(\frac{p_1}{p_2}\right)_4 \left(\frac{p_2}{p_1}\right)_4 = \left(\frac{p_1}{ac+bd}\right)$, where $p_1 = a^2 + b^2$ and $p_2 = c^2 + d^2$; on the other hand, according to [11] we have $\left(\frac{p_1}{ac+bd}\right) = \left(\frac{\pi_1}{\pi_3}\right)$, which implies the result. \square

Lemma 3. *Put $\mathbb{k} = \mathbb{Q}(\sqrt{2p_1 p_2}, i)$ and $\mathbb{K}_3 = \mathbb{Q}(\sqrt{2}, \sqrt{p_1 p_2}, i)$. Then*

- (1) $\{\varepsilon_{2p_1 p_2}\}$ is a FSU of \mathbb{k} .
- (2) *If $N(\varepsilon_{p_1 p_2}) = 1$ or $N(\varepsilon_{p_1 p_2}) = -1$ and $\sqrt{\varepsilon_2 \varepsilon_{p_1 p_2} \varepsilon_{2p_1 p_2}} \notin \mathbb{K}_3^+$, then*
 - (i) $\{\varepsilon_2, \varepsilon_{p_1 p_2}, \varepsilon_{2p_1 p_2}\}$ is a FSU of both \mathbb{K}_3^+ and \mathbb{K}_3 .
 - (ii) $q = 1$, $q(\mathbb{K}_3/\mathbb{Q}) = 2$ and $h(\mathbb{K}_3) = h(p_1 p_2)h(-p_1 p_2)$.
- (3) *If $N(\varepsilon_{p_1 p_2}) = -1$ and $\sqrt{\varepsilon_2 \varepsilon_{p_1 p_2} \varepsilon_{2p_1 p_2}} \in \mathbb{K}_3^+$, then*
 - (i) $\{\varepsilon_2, \varepsilon_{p_1 p_2}, \sqrt{\varepsilon_2 \varepsilon_{p_1 p_2} \varepsilon_{2p_1 p_2}}\}$ is a FSU of both \mathbb{K}_3^+ and \mathbb{K}_3 .
 - (ii) $q = 2$, $q(\mathbb{K}_3/\mathbb{Q}) = 4$ and $h(\mathbb{K}_3) = 2h(p_1 p_2)h(-p_1 p_2)$.

Proof. (1) As $N(\varepsilon_{2p_1 p_2}) = -1$, so if $N(\varepsilon_{p_1 p_2}) = -1$ we get, according to [1, Applications 1), p.114], that $\{\varepsilon_{2p_1 p_2}\}$ is a FSU of \mathbb{k} .

(2) Assume that $N(\varepsilon_{p_1 p_2}) = 1$. As $N(\varepsilon_2) = N(\varepsilon_{2p_1 p_2}) = -1$, so $\varepsilon_2, \varepsilon_{2p_1 p_2}, \varepsilon_2 \varepsilon_{p_1 p_2}, \varepsilon_2 \varepsilon_{2p_1 p_2}, \varepsilon_{p_1 p_2} \varepsilon_{2p_1 p_2}$ and $\varepsilon_2 \varepsilon_{p_1 p_2} \varepsilon_{2p_1 p_2}$ are not squares in \mathbb{K}_3^+ , else by taking a suitable norm we get $i \in \mathbb{K}_3^+$, which is false. Furthermore $(2 + \sqrt{2})\varepsilon_2^i \varepsilon_{p_1 p_2}^j \varepsilon_{2p_1 p_2}^k$ can not be a square in \mathbb{K}_3^+ , for all i, j and k in $\{0, 1\}$, as otherwise with some $\alpha \in \mathbb{K}_3^+$ we would have $\alpha^2 = (2 + \sqrt{2})\varepsilon_2^i \varepsilon_{p_1 p_2}^j \varepsilon_{2p_1 p_2}^k$, so $(N_{\mathbb{K}^+/\mathbb{Q}(\sqrt{p_1 p_2})}(\alpha))^2 = 2(-1)^{i+k} \varepsilon_{p_1 p_2}^{2j}$, yielding that $\sqrt{2} \in \mathbb{Q}(\sqrt{p_1 p_2})$, which is absurd.

Put $\varepsilon_{p_1 p_2} = a + b\sqrt{p_1 p_2}$; as $2p_1(a \pm 1)$ is a square in \mathbb{N} , thus $\sqrt{2\varepsilon_{p_1 p_2}} = b_1\sqrt{2p_1} + b_2\sqrt{2p_2}$, where $b_i \in \mathbb{Z}$; so $\varepsilon_{p_1 p_2}$ is not a square in \mathbb{K}_3^+ ; hence $\{\varepsilon_2, \varepsilon_{p_1 p_2}, \varepsilon_{2p_1 p_2}\}$ is a FSU of \mathbb{K}_3^+ , which implies that $q = 1$. Thus from [1, Proposition 3, p.112] we get $\{\varepsilon_2, \varepsilon_{p_1 p_2}, \varepsilon_{p_1 p_2 q}\}$ is a FSU of \mathbb{K}_3 , we infer that $q(\mathbb{K}_3/\mathbb{Q}) = 2$, since $\sqrt{i} \in \mathbb{K}_3$. If $N(\varepsilon_{p_1 p_2}) = -1$, then the results are guaranteed by [1, Propositions 8, 15]. In the end, under our conditions, P.Kaplan states in [17] that $h(2p_1 p_2) = h(-2p_1 p_2) = 4$, therefore the class number formula yields that $h(\mathbb{K}_3^+) = h(p_1 p_2)$ and $h(\mathbb{K}_3) = h(p_1 p_2)h(-p_1 p_2)$.

(3) If $N(\varepsilon_{p_1 p_2}) = -1$ and $\sqrt{\varepsilon_2 \varepsilon_{p_1 p_2} \varepsilon_{2p_1 p_2}} \in \mathbb{K}_3^+$, then the results are also deduced from [1, Propositions 8, 15] and the class number formula implies (3)(iii). \square

Lemma 4. *Let $\kappa_{\mathbb{K}_3}$ denote the set of classes of $\mathbf{Cl}_2(\mathbb{k})$ that capitulate in \mathbb{K}_3 , then*

$$\kappa_{\mathbb{K}_3} = \begin{cases} \langle [\mathcal{H}_0] \rangle & \text{if } q = 2, \\ \langle [\mathcal{H}_0], [\mathcal{H}_1 \mathcal{H}_2] \rangle & \text{if } q = 1. \end{cases}$$

Proof. From Lemma 3 we get $E_{\mathbb{k}} = \langle i, \varepsilon_{2p_1 p_2} \rangle$ and $E_{\mathbb{K}_3} = \langle \sqrt{i}, \varepsilon_2, \varepsilon_{p_1 p_2}, \varepsilon_{2p_1 p_2} \rangle$ or $E_{\mathbb{K}_3} = \langle \sqrt{i}, \varepsilon_2, \varepsilon_{p_1 p_2}, \sqrt{\varepsilon_2 \varepsilon_{p_1 p_2} \varepsilon_{p_1 p_2 q}} \rangle$, according as $q = 1$ or 2 . Therefore $N_{\mathbb{K}_3/\mathbb{k}}(E_{\mathbb{K}_3}) = \langle i, \varepsilon_{2p_1 p_2}^2 \rangle$ or $\langle i, \varepsilon_{2p_1 p_2} \rangle$, thus

$$[E_{\mathbb{k}} : N_{\mathbb{K}_3/\mathbb{k}}(E_{\mathbb{K}_3})] = \begin{cases} 2, & \text{if } q = 1, \\ 1, & \text{if } q = 2; \end{cases} \quad \text{hence } \#\kappa_{\mathbb{K}_3} = \begin{cases} 4, & \text{if } q = 1, \\ 2, & \text{if } q = 2. \end{cases}$$

Moreover, it is easy to see that $\sqrt{(1+i)\varepsilon_2} = \frac{1}{2}(2 + (1+i)\sqrt{2})$, so there exists $\beta \in \mathbb{K}_3$ such that $\mathcal{H}_0^2 = (1+i) = (\beta^2)$, this implies that \mathcal{H}_0 capitulates in \mathbb{K}_3 . Consequently, if $q = 2$, then $\kappa_{\mathbb{K}_3} = \langle [\mathcal{H}_0] \rangle$.

Suppose $q = 1$, then we have two cases to discuss:

(a) If $N(\varepsilon_{p_1 p_2}) = 1$, then by Lemma 1, we get $\sqrt{2\varepsilon_{p_1 p_2}} = b_1\sqrt{2p_1} + b_2\sqrt{2p_2}$, where $b = 2b_1 b_2$, from which we deduce that $2p_1 \varepsilon_{p_1 p_2}$ is a square in \mathbb{K}_3 , thus there exists $\alpha \in \mathbb{K}_3$ such that $(2p_1) = (\alpha^2)$. On the other hand, $(\mathcal{H}_1 \mathcal{H}_2)^2 = (p_1)$ and $(2) = (2i) = (1+i)^2$, hence $\mathcal{H}_1 \mathcal{H}_2 = (\frac{\alpha}{1+i})$, which implies that $\mathcal{H}_1 \mathcal{H}_2$ capitulates in \mathbb{K}_3 .

(b) If $N(\varepsilon_{p_1 p_2}) = -1$, then there exist an even integer a and an odd integer b such that $\varepsilon_{p_1 p_2} = a + b\sqrt{p_1 p_2}$, so $a^2 + 1 = b^2 p_1 p_2$, since $p_1 p_2 \equiv 1 \pmod{8}$. Therefore:

$$\begin{cases} a \mp i = ib_1^2 \pi_1 \pi_3, \\ a \pm i = -ib_2^2 \pi_2 \pi_4, \end{cases} \quad \text{or} \quad \begin{cases} a \mp i = ib_1^2 \pi_1 \pi_4, \\ a \pm i = -ib_2^2 \pi_2 \pi_3, \end{cases}$$

$$\text{hence } \left. \begin{aligned} \sqrt{\varepsilon_{p_1 p_2}} &= z_1 \sqrt{\pi_1 \pi_3} + z_2 \sqrt{\pi_2 \pi_4} \text{ or } \\ \sqrt{\varepsilon_{p_1 p_2}} &= z_1 \sqrt{\pi_1 \pi_4} + z_2 \sqrt{\pi_2 \pi_3}, \end{aligned} \right\} \quad (1)$$

where z_2 is the conjugate of z_1 in $\frac{1}{2}\mathbb{Z}[i]$.

Similarly, as $N(\varepsilon_{2p_1 p_2}) = -1$, so there exist x, y in \mathbb{N} such that $x^2 + 1 = 2p_1 p_2 y^2$, and

$$\left. \begin{aligned} \sqrt{\varepsilon_{2p_1 p_2}} &= y_1 \sqrt{(1+i)\pi_1 \pi_3} + y_2 \sqrt{(1-i)\pi_2 \pi_4}, \text{ or } \\ \sqrt{\varepsilon_{2p_1 p_2}} &= y_1 \sqrt{(1+i)\pi_1 \pi_4} + y_2 \sqrt{(1-i)\pi_2 \pi_3}, \text{ or } \\ \sqrt{2\varepsilon_{2p_1 p_2}} &= y_1 \sqrt{(1+i)\pi_1 \pi_3} + y_2 \sqrt{(1-i)\pi_2 \pi_4}, \text{ or } \\ \sqrt{2\varepsilon_{2p_1 p_2}} &= y_1 \sqrt{(1+i)\pi_1 \pi_3} + y_2 \sqrt{(1-i)\pi_2 \pi_4}, \end{aligned} \right\} \quad (2)$$

where y_i are in $\mathbb{Z}[i]$ or $\frac{1}{2}\mathbb{Z}[i]$.

Finally, Note that:

$$\sqrt{2\varepsilon_1} = \sqrt{1+i} + \sqrt{1-i}. \quad (3)$$

So by multiplying the equalities (1), (2) and (3) we get

$$\begin{aligned} \sqrt{\varepsilon_1 \varepsilon_2 \varepsilon_3} &= \alpha + \beta \sqrt{2} + \gamma \sqrt{p_1 p_2} + \delta \sqrt{2p_1 p_2} \in \mathbb{Q}(\sqrt{2}, \sqrt{p_1 p_2}) \text{ or } \\ \sqrt{\varepsilon_1 \varepsilon_2 \varepsilon_3} &= \alpha \sqrt{p_1} + \beta \sqrt{p_2} + \gamma \sqrt{2p_1} + \delta \sqrt{2p_2} \notin \mathbb{Q}(\sqrt{2}, \sqrt{p_1 p_2}), \end{aligned}$$

where α, β, γ and δ are in \mathbb{Q} .

As $q = 1$, so $\varepsilon_2 \varepsilon_{p_1 p_2} \varepsilon_{p_1 p_2 q}$ is not a square in \mathbb{K}_3^+ , hence $p_1 \varepsilon_2 \varepsilon_{p_1 p_2} \varepsilon_{p_1 p_2 q}$ is a square in \mathbb{K}_3 ; which yields that $\mathcal{H}_1 \mathcal{H}_2$ capitulates in \mathbb{K}_3 . Thus $\kappa_{\mathbb{K}_3} = \langle [\mathcal{H}_0], [\mathcal{H}_1 \mathcal{H}_2] \rangle$. \square

Proposition 1 ([4]). *Let $p_1 \equiv p_2 \equiv 1 \pmod{4}$ be different primes such that $\left(\frac{2}{p_1}\right) = \left(\frac{2}{p_2}\right) = \left(\frac{p_1}{p_2}\right)$. Then*

$$\left(\frac{p_1 p_2}{2}\right)_4 \left(\frac{2p_1}{p_2}\right)_4 \left(\frac{2p_2}{p_1}\right)_4 = \left(\frac{\pi_1}{\pi_3}\right) \left(\frac{1+i}{\pi_1}\right) \left(\frac{1+i}{\pi_3}\right).$$

Proposition 2 ([4]). *Let $p_1 \equiv p_2 \equiv 1 \pmod{4}$ be different primes such that $\left(\frac{2}{p_1}\right) = \left(\frac{2}{p_2}\right) = \left(\frac{p_1}{p_2}\right) = -1$. Then the following assertions are equivalent:*

- (1) $\varepsilon_2 \varepsilon_{p_1 p_2} \varepsilon_{2p_1 p_2}$ is a square in \mathbb{K}_3^+ .
- (2) $\left(\frac{p_1 p_2}{2}\right)_4 \left(\frac{2p_1}{p_2}\right)_4 \left(\frac{2p_2}{p_1}\right)_4 = -1$.
- (3) $q(\mathbb{K}_3^+/\mathbb{Q}) = 2$.

The following results are deduced from [9].

Theorem 4. *Let $p_1 \equiv p_2 \equiv 5 \pmod{8}$ be different primes and put $F_1 = \mathbb{Q}(\sqrt{p_1 p_2}, i)$.*

- (1) $\mathbf{Cl}_2(\bar{k}_1)$ is of type $(2, 2^m)$, $m \geq 2$. It is generated by $2 = (2, 1 + \sqrt{-p_1 p_2})$, the prime ideal of \bar{k}_1 above 2, and an ideal I of \bar{k}_1 of order 2^m . Moreover

$$\begin{cases} I^{2^{m-1}} \sim \mathfrak{p}_1 & \text{if } \left(\frac{p_1}{p_2}\right) = 1, \\ I^{2^{m-1}} \sim 2\mathfrak{p}_1 & \text{if } \left(\frac{p_1}{p_2}\right) = -1; \end{cases}$$

where $\mathfrak{p}_1 = (p_1, \sqrt{-p_1 p_2})$ is the prime ideal of \bar{k}_1 above p_1 .

- (2) $\mathbf{Cl}_2(k_1)$ is of type (2^n) , $n \geq 1$, and it is generated by 2_1 , a prime ideal of k_1 above 2.
- (3) $\mathbf{Cl}_2(F_1)$ is of 2-rank equal to 2. It is generated by I and 2_{F_1} , where 2_{F_1} is a prime ideal of F_1 above 2.
- (4) If $\left(\frac{p_1}{p_2}\right) = -1$, then $\mathbf{Cl}_2(F_1) \simeq (2^n, 2^m)$; and, in $\mathbf{Cl}_2(F_1)$, $I^{2^{m-1}} \sim 2_{F_1}^{2^n} \sim \mathfrak{p}_1 \not\sim 1$.
- (5) If $\left(\frac{p_1}{p_2}\right) = 1$ and $N(\varepsilon_{p_1 p_2}) = -1$, then

$$\mathbf{Cl}_2(F_1) \simeq (2^{\min(n, m-1)}, 2^{\max(m-1, n+1)})$$

and $I^{2^{m-1}} \sim 2_{F_1}^{2^n} \sim \mathfrak{p}_1 \not\sim 1$.

- (6) If $\left(\frac{p_1}{p_2}\right) = 1$ and $N(\varepsilon_{p_1 p_2}) = 1$, then $\mathbf{Cl}_2(F_1) \simeq (2^{n+1}, 2^{m-1})$; moreover $I^{2^{m-1}} \sim 2_{F_1}^{2^{n+1}} \sim \mathfrak{p}_1 \sim 1$.

Using the above theorem, we prove the following lemma.

Lemma 5. Let $\mathfrak{p}_1 \mathcal{O}_{F_1} = \mathcal{P}_1 \mathcal{P}_2$ and $p_2 \mathcal{O}_{F_1} = \mathcal{P}_3^2 \mathcal{P}_4^2$, then in $\mathbf{Cl}_2(F_1)$ we have:

- (i) If $\left(\frac{p_1}{p_2}\right) = -1$ or $\left(\frac{p_1}{p_2}\right) = 1$ and $N(\varepsilon_{p_1 p_2}) = -1$, then $\mathcal{P}_1 \sim 2_{F_1}^{2^{n-1}} I^{2^{m-2}}$.
- (ii) If $\left(\frac{p_1}{p_2}\right) = 1$ and $N(\varepsilon_{p_1 p_2}) = 1$, then $\mathcal{P}_1 \sim 2_{F_1}^{2^n} I^{2^{m-2}}$ or $\mathcal{P}_1 \sim I^{2^{m-2}}$.
Moreover $\mathcal{P}_1 \mathcal{P}_3 \sim 2_{F_1}^{2^n}$.

Proof. Let $p_1 \mathcal{O}_{\mathbb{Q}(i)} = \pi_1 \pi_2$, $p_2 \mathcal{O}_{\mathbb{Q}(i)} = \pi_3 \pi_4$, $\mathfrak{p}_1 \mathcal{O}_{F_1} = \mathcal{P}_1 \mathcal{P}_2$ and $\mathfrak{p}_2 \mathcal{O}_{F_1} = \mathcal{P}_3 \mathcal{P}_4$, where \mathfrak{p}_2 is the prime ideal of \bar{k}_1 above p_2 , then $(\pi_i) = \mathcal{P}_i^2$, for all i . So, according to [3, Proposition 1], \mathcal{P}_i are not principals in F_1 and they are of order two. On the other hand, as the 2-rank of $\mathbf{Cl}_2(F_1)$ is 2, thus $\mathcal{P}_i \in \langle [2_{F_1}], [I] \rangle$.

(i) In this case, we have $\mathfrak{p}_1 \not\sim 1$, hence $\mathcal{P}_1 \not\sim \mathcal{P}_2$; note that the elements of order two in $\mathbf{Cl}_2(F_1)$ are $2_{F_1}^{2^{n-1}} I^{2^{m-2}}$, $2_{F_1}^{2^{n-1}} I^{-2^{m-2}}$ and $2_{F_1}^{2^n} \sim I^{2^{m-1}}$. Therefore \mathcal{P}_1 is equivalent to one of these three elements. As $\mathcal{P}_1 \sim 2_{F_1}^{2^n} \sim I^{2^{m-1}}$ can not occur, if not we would have, by applying the norm N_{F_1/\bar{k}_1} , $\mathfrak{p}_1 \sim I^{2^m} \sim 1$, which is false. Thus $\mathcal{P}_1 \sim 2_{F_1}^{2^{n-1}} I^{2^{m-2}}$ and $\mathcal{P}_2 \sim 2_{F_1}^{2^{n-1}} I^{-2^{m-2}}$ or $\mathcal{P}_1 \sim 2_{F_1}^{2^{n-1}} I^{-2^{m-2}}$ and

$\mathcal{P}_2 \sim 2_{F_1}^{2^{n-1}} I^{2^{m-2}}$. Hence with out loss of generality we can choose $\mathcal{P}_1 \sim 2_{F_1}^{2^{n-1}} I^{2^{m-2}}$.

(ii) In this case, we have $\mathfrak{p}_1 \sim \mathfrak{p}_2 \sim 1$, hence $\mathcal{P}_1 \sim \mathcal{P}_2$ and $\mathcal{P}_3 \sim \mathcal{P}_4$. On the other hand, according to [3, Proposition 1], $\mathcal{P}_1 \mathcal{P}_3$ is not principal in F_1 . To this end, note that the elements of order two in $\mathbf{Cl}_2(F_1)$ are $2_{F_1}^{2^n} I^{2^{m-2}}$, $2_{F_1}^{2^n}$ and $I^{-2^{m-2}}$. Therefore \mathcal{P}_1 is equivalent to one of these three elements. As $\mathcal{P}_1 \sim 2_{F_1}^{2^n}$ can not occur, as otherwise, by applying the norm N_{F_1/\bar{k}_1} , we get $\mathfrak{p}_1 \sim 2^{2^n} \sim 1$, which is false. Thus $\mathcal{P}_1 \sim I^{2^{m-2}}$ and $\mathcal{P}_3 \sim 2_{F_1}^{2^n} I^{2^{m-2}}$ or $\mathcal{P}_1 \sim 2_{F_1}^{2^{n-1}} I^{2^{m-2}}$ and $\mathcal{P}_3 \sim I^{2^{m-2}}$. Hence $\mathcal{P}_1 \mathcal{P}_3 \sim 2_{F_1}^{2^n}$. \square

We conclude this section with the following lemma which gives the relationship between the unit index q and the integers n and m . It is a consequence of Lemma 3, Proposition 2 and the results in [6], [17].

Lemma 6. (1) Suppose $q = 1$, so

- (i) If $\left(\frac{p_1}{p_2}\right) = -1$, then $n = 1$ and $m \geq 3$.
 - (ii) If $\left(\frac{p_1}{p_2}\right) = 1$, then
 - (a) If $\left(\frac{p_1}{p_2}\right)_4 \left(\frac{p_2}{p_1}\right)_4 = -1$, then $n = 1$ and $m \geq 3$.
 - (b) If $\left(\frac{p_1}{p_2}\right)_4 \left(\frac{p_2}{p_1}\right)_4 = 1$, then $m = 2$ and $n \geq 2$.
- (2) Suppose $q = 2$, so
- (i) If $\left(\frac{p_1}{p_2}\right) = -1$, then $n = 1$ and $m = 2$.
 - (ii) If $\left(\frac{p_1}{p_2}\right) = 1$, then $m = 2$ and $n \geq 2$.

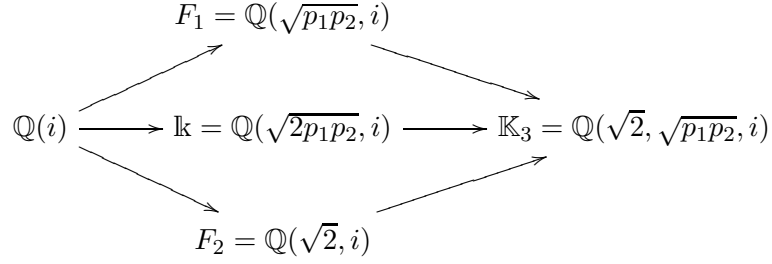
4. PROOFS OF THE MAIN RESULTS

Recall first the following result from [12, p. 205].

Lemma 7. If \mathcal{H} is an unramified ideal in some extension $\mathbb{K}/\mathbb{k} = \mathbb{k}(\sqrt{x})/\mathbb{k}$, then the quadratic residue symbol is given by the Artin symbol $\varphi = \left(\frac{\mathbb{k}(\sqrt{x})/\mathbb{k}}{\mathcal{H}}\right)$ as follows: $\left(\frac{x}{\mathcal{H}}\right) = \sqrt{x}^{\varphi-1}$.

4.1. Proof of Theorem 2. (1) The assertion $\mathbf{Cl}_2(\mathbb{k}) = \langle [\mathcal{H}_0], [\mathcal{H}_1], [\mathcal{H}_2] \rangle \simeq (2, 2, 2)$ of Theorem 2 is proved in [5] and [3]. In the following pages we will prove the other assertions.

(2) To prove the second assertion we will use the techniques that F. Lemmermeyer has used in some of his works see for example [9] or [10]. Consider the following diagram

FIGURE 1. Subfields of $\mathbb{K}_3/\mathbb{Q}(i)$

Compute first $N_{\mathbb{K}_3/\mathbb{k}}(\mathbf{Cl}_2(\mathbb{K}_3))$. Recall that

$$N_{\mathbb{K}_3/\mathbb{k}}(\mathbf{Cl}_2(\mathbb{K}_3)) = \{[\mathcal{H}] \in \mathbf{Cl}_2(\mathbb{k}) / \left(\frac{2}{[\mathcal{H}]}\right) = 1\}.$$

As \mathcal{H}_1 and \mathcal{H}_2 are unramified in $\mathbb{K}_3/\mathbb{k} = \mathbb{k}(\sqrt{2})/\mathbb{k} = \mathbb{k}(\sqrt{p_1 p_2})/\mathbb{k}$, so Lemma 7 yields that $\left(\frac{2}{\mathcal{H}_1 \mathcal{H}_2}\right) = \left(\frac{2}{\mathcal{H}_1}\right) \left(\frac{2}{\mathcal{H}_2}\right) = \left(\frac{2}{p_1}\right) \left(\frac{2}{p_1}\right) = 1$. On the other hand, 2 ramifies completely in \mathbb{k}/\mathbb{Q} and splits in F_1/\mathbb{Q} ; moreover \mathcal{H}_0 is unramified in \mathbb{K}_3/\mathbb{k} , then \mathcal{H}_0 splits in \mathbb{K}_3 i.e. $\mathcal{H}_0 \in N_{\mathbb{K}_3/\mathbb{k}}(\mathbf{Cl}_2(\mathbb{K}_3))$. Thus

$$N_{\mathbb{K}_3/\mathbb{k}}(\mathbf{Cl}_2(\mathbb{K}_3)) = \langle [\mathcal{H}_0], [\mathcal{H}_1 \mathcal{H}_2] \rangle.$$

To this end, it is easy to see that \mathbb{K}_3/F_1 and \mathbb{K}_3/F_2 are ramified extensions, whereas \mathbb{K}_3/\mathbb{k} is not; so from the class field theory $[\mathbf{Cl}_2(\mathbb{k}) : N_{\mathbb{K}_3/\mathbb{k}}(\mathbf{Cl}_2(\mathbb{K}_3))] = 2$, $\mathbf{Cl}_2(F_2) = N_{\mathbb{K}_3/F_2}(\mathbf{Cl}_2(\mathbb{K}_3))$ and $\mathbf{Cl}_2(F_1) = N_{\mathbb{K}_3/F_1}(\mathbf{Cl}_2(\mathbb{K}_3))$, hence Theorem 4 implies that

$$N_{\mathbb{K}_3/F_1}(\mathbf{Cl}_2(\mathbb{K}_3)) = \langle [2_{F_1}], [I] \rangle.$$

Then there exists an ideal $\mathfrak{P} \in \mathbb{K}_3$ such that $N_{\mathbb{K}_3/F_1}(\mathfrak{P}) \sim I$ and $N_{\mathbb{K}_3/\mathbb{k}}(\mathfrak{P}) \in \langle [\mathcal{H}_0], [\mathcal{H}_1 \mathcal{H}_2] \rangle$. One shows that $N_{\mathbb{K}_3/\mathbb{k}}(\mathfrak{P}) \sim \mathcal{H}_1 \mathcal{H}_2$ (see Lemma 8 below). We claim that

$$\begin{cases} \mathfrak{P}^2 \sim I, & \text{if } q = 1, \\ \mathfrak{P}^2 \sim \mathcal{H}_1 \mathcal{H}_2 I, & \text{if } q = 2. \end{cases}$$

Let t and s be the elements of order 2 of $\text{Gal}(\mathbb{K}_3/\mathbb{Q}(i))$ which fix F_1 and \mathbb{k} , respectively. Using the identity $2 + (1 + t + s + ts) = (1 + t) + (1 + s) + (1 + ts)$ of the group ring $\mathbb{Z}[\text{Gal}(\mathbb{K}_3/\mathbb{Q}(i))]$ and observing that the class numbers of $\mathbb{Q}(i)$, F_2 are odd, we find that

$$\mathfrak{P}^2 \sim \mathfrak{P}^{1+t} \mathfrak{P}^{1+s} \mathfrak{P}^{1+ts} \sim \mathcal{H}_1 \mathcal{H}_2 I.$$

As $\mathcal{H}_1\mathcal{H}_2 \sim 1$, in $\mathbf{Cl}_2(\mathbb{K}_3)$, if $q = 1$ (see Lemma 4), so the result claimed. Moreover $\mathfrak{p}_1\mathcal{O}_{\mathbb{K}} = \mathcal{H}_1\mathcal{H}_2$, so Lemma 4 yields that in $\mathbf{Cl}_2(\mathbb{K}_3)$ we have

$$\begin{cases} \mathfrak{P}^{2^m} \sim I^{2^{m-1}} \sim \mathfrak{p}_1 \sim 1, & \text{if } q = 1, \\ \mathfrak{P}^{2^m} \sim I^{2^{m-1}} \sim \mathfrak{p}_1, & \text{if } q = 2. \end{cases}$$

On the other hand, 2_{F_1} ramifies and \mathcal{H}_0 splits in \mathbb{K}_3 , let \mathfrak{A} be an ideal of \mathbb{K}_3 above 2_{F_1} , then $N_{\mathbb{K}_3/\mathbb{K}}(\mathfrak{A}) \sim \mathcal{H}_0$ and $N_{\mathbb{K}_3/F_1}(\mathfrak{A}) \sim 2_{F_1}$. Thus, in $\mathbf{Cl}_2(\mathbb{K}_3)$, we have

$$\mathfrak{A}^2 \sim 2_{F_1} \text{ and } \mathfrak{A}^{2^{n+1}} \sim 2_{F_1}^{2^n}.$$

Recall that \mathcal{H}_j and \mathcal{P}_j coincide and remain inert in \mathbb{K}_3 ; moreover $\mathfrak{p}_1\mathcal{O}_{\mathbb{K}_3} = \mathcal{H}_1\mathcal{H}_2\mathcal{O}_{\mathbb{K}_3} = \mathcal{P}_1\mathcal{P}_2\mathcal{O}_{\mathbb{K}_3}$ and $\mathfrak{p}_2\mathcal{O}_{\mathbb{K}_3} = \mathcal{H}_3\mathcal{H}_4\mathcal{O}_{\mathbb{K}_3} = \mathcal{P}_3\mathcal{P}_4\mathcal{O}_{\mathbb{K}_3}$, therefore:

- If $\left(\frac{p_1}{p_2}\right) = -1$ or $\left(\frac{p_1}{p_2}\right) = 1$ and $N(\varepsilon_{p_1p_2}) = -1$, then Lemmas 3, 4 and Theorem 4 imply that

$$\begin{cases} \mathfrak{A}^{2^{n+1}} \sim 2_{F_1}^{2^n} \sim 1, & \text{if } q = 1, \\ \mathfrak{A}^{2^{n+1}} \sim 2_{F_1}^{2^n} \not\sim 1 \text{ and } \mathfrak{A}^{2^{n+2}} \sim 2_{F_1}^{2^{n+1}} \sim 1, & \text{if } q = 2. \end{cases}$$

- If $\left(\frac{p_1}{p_2}\right) = 1$ and $N(\varepsilon_{p_1p_2}) = 1$, then $q = 1$, and Lemma 5 yields that $\mathcal{P}_1\mathcal{P}_3 \sim 2_{F_1}^{2^n}$.

Let us prove that $\mathcal{P}_1\mathcal{P}_3\mathcal{O}_{\mathbb{K}_3} = \mathcal{H}_1\mathcal{H}_3\mathcal{O}_{\mathbb{K}_3}$ is principal. We know that $N(\varepsilon_{2p_1p_2}) = -1$, so the decomposition uniqueness in $\mathbb{Z}[i]$ implies that there exist y_1, y_2 in $\mathbb{Z}[i]$ such that

$$\begin{aligned} \sqrt{\varepsilon_d} &= \frac{1}{2}[y_1(1+i)\sqrt{(1\pm i)\pi_1\pi_3} + y_2(1-i)\sqrt{(1\mp i)\pi_2\pi_4}] \text{ (a) or} \\ \sqrt{\varepsilon_d} &= \frac{1}{2}[y_1(1+i)\sqrt{(1\pm i)\pi_1\pi_4} + y_2(1-i)\sqrt{(1\mp i)\pi_2\pi_3}] \text{ (b).} \end{aligned}$$

Moreover the ideal $\mathcal{H}_0\mathcal{H}_1\mathcal{H}_3$ is principal in \mathbb{K} , if and only if there exists a unit $\varepsilon \in \mathbb{K}$ such that

$$(1+i)\pi_1\pi_3\varepsilon = \alpha^2, \tag{4}$$

where $\alpha \in \mathbb{K}$. As $N(\varepsilon_{2p_1p_2}) = -1$, so Lemma 3 involves that $Q_{\mathbb{K}} = 1$, the unit index of \mathbb{K} ; hence ε is either real or purely imaginary.

Put $\alpha = \alpha_1 + i\alpha_2$, with $\alpha_1, \alpha_2 \in \mathbb{Q}(\sqrt{2p_1p_2})$, and suppose ε is real (same proof if it is purely imaginary); as $\pi_1\pi_3 = (e + 2if)(g + 2ih) = (eg - 4fh) + 2i(eh + gf)$, so the equation (4) is equivalent to

$$\alpha_1^2 - \alpha_2^2 + 2i\alpha_1\alpha_2 = \varepsilon[(eg - 4fh) - 2(eh + fg)] + i\varepsilon_d[(eg - 4fh) + 2(eh + fg)],$$

hence

$$\begin{cases} \alpha_1^2 - \alpha_2^2 &= \varepsilon[(eg - 4fh) - 2(eh + fg)], \\ 2\alpha_1\alpha_2 &= \varepsilon[(eg - 4fh) + 2(eh + fg)], \end{cases}$$

so we get $\alpha_2 = \frac{\varepsilon[(eg-4fh)+2(eh+gf)]}{2\alpha_1}$, thus

$$4\alpha_1^4 - 4\varepsilon[(eg-4fh) - 2(eh+fg)]\alpha_1^2 - [(eg-4fh) + 2(eh+fg)]^2\varepsilon^2 = 0,$$

the discriminant of this equation is $\Delta' = 4\varepsilon^2d$, $d = 2p_1p_2$, which implies that

$$\alpha_1^2 = \frac{\varepsilon}{4}[2[(eg-4fh) - 2(eh+fg)] \pm 2\sqrt{d}].$$

Since

$$(1+i)\pi_1\pi_3 + (1-i)\pi_2\pi_4 = 2(eg-4fh) - 4(eh+fg) \text{ and } \\ \sqrt{d} = \sqrt{(1-i)\pi_1\pi_3}\sqrt{(1+i)\pi_2\pi_4},$$

then

$$\begin{aligned} \alpha_1^2 &= \frac{\varepsilon}{4}(\sqrt{(1-i)\pi_1\pi_3} + \sqrt{(1+i)\pi_2\pi_4})^2, \text{ so} \\ \alpha_1 &= \frac{\sqrt{\varepsilon}}{2}(\sqrt{(1-i)\pi_1\pi_3} + \sqrt{(1+i)\pi_2\pi_4}), \end{aligned}$$

therefore if $\varepsilon = \varepsilon_d$ and $\sqrt{\varepsilon_d}$ takes the value (a), we get

$$\alpha_1 = \frac{1}{4}(2y_1\pi_1\pi_3 + 2y_2\pi_2\pi_4 + (y_1(1+i) + y_2(1-i))\sqrt{d}),$$

and

$$\alpha_2 = \frac{\varepsilon_d[(eg-4fh) + 2(eh+gf)]}{2\alpha_1},$$

and it is easy to see that $\alpha_1, \alpha_2 \in \mathbb{Q}(\sqrt{2p_1p_2})$; hence $\mathcal{H}_0\mathcal{H}_1\mathcal{H}_3$ is principal in \mathbb{k} . Proceeding similarly, we prove that $\mathcal{H}_0\mathcal{H}_2\mathcal{H}_3$ is principal in \mathbb{k} if $\sqrt{\varepsilon_d}$ takes the value (b). Hence, in $\mathbf{Cl}_2(\mathbb{k})$, we have $\mathcal{H}_3 \sim \mathcal{H}_0\mathcal{H}_1$ or $\mathcal{H}_3 \sim \mathcal{H}_0\mathcal{H}_2$, this in turn shows that, in $\mathbf{Cl}_2(\mathbb{K}_3)$, $\mathcal{H}_1\mathcal{H}_3 \sim \mathcal{H}_0\mathcal{H}_1\mathcal{H}_2$ or $\mathcal{H}_1\mathcal{H}_3 \sim \mathcal{H}_0$, as we know that $\mathcal{H}_0, \mathcal{H}_1\mathcal{H}_2$ capitulate in \mathbb{K}_3 , so the result. Thus

$$\mathfrak{A}^{2^{n+1}} \sim 2_{F_1}^{2^n} \sim 1.$$

Consequently, in $\mathbf{Cl}_2(\mathbb{K}_3)$, we have

$$\begin{cases} \mathfrak{A}^{2^{n+1}} \sim \mathfrak{P}^{2^m} \sim \mathcal{H}_1\mathcal{H}_2 \sim 1, & \text{if } q = 1, \\ \mathfrak{A}^{2^{n+1}} \sim \mathfrak{P}^{2^m} \sim \mathcal{H}_1\mathcal{H}_2 \not\sim 1 \text{ and } \mathfrak{A}^{2^{n+2}} \sim \mathfrak{P}^{2^{m+1}} \sim 1, & \text{if } q = 2. \end{cases}$$

To this end, note that for all $i \leq n, j \leq m-1$, we have $\mathfrak{A}^{2^i}\mathfrak{P}^{2^j} \not\sim 1$, as otherwise, we would have, by applying the norm $N_{\mathbb{K}_3/F_1}$, $2_{F_1}^{2^i}I^{2^j} \sim 1$, which contradicts the results of Theorem 4.

Conclusion

If $q = 1$, then $\langle [\mathfrak{A}], [\mathfrak{P}] \rangle$ is a subgroup of $\mathbf{Cl}_2(\mathbb{K}_3)$ of type $(2^m, 2^{n+1})$, and as in this case $h(\mathbb{K}_3) = h(p_1 p_2) h(-p_1 p_2) = 2^{n+m+1}$ (see Lemma 3), so

$$\mathbf{Cl}_2(\mathbb{K}_3) = \langle [\mathfrak{A}], [\mathfrak{P}] \rangle \simeq (2^{n+1}, 2^m).$$

If $q = 2$, then $\langle [\mathfrak{A}], [\mathfrak{P}] \rangle$ is a subgroup of $\mathbf{Cl}_2(\mathbb{K}_3)$ of type $(2^{\min(m, n+1)}, 2^{\max(n+2, m+1)})$, and as in this case $h(\mathbb{K}_3) = 2h(p_1 p_2) h(-p_1 p_2) = 2^{n+m+2}$ (see Lemma 3), so

$$\mathbf{Cl}_2(\mathbb{K}_3) = \langle [\mathfrak{A}], [\mathfrak{P}] \rangle \simeq (2^{\min(m, n+1)}, 2^{\max(n+2, m+1)}).$$

As \mathcal{H}_1 , \mathcal{P}_1 remain inert in \mathbb{K}_3 , so they do not capitulate and coincide in \mathbb{K}_3 , hence they are of order 2. Thus from Theorem 4 and Lemma 5 we deduce that:

- If $\left(\frac{p_1}{p_2}\right) = -1$ or $\left(\frac{p_1}{p_2}\right) = 1$ and $N(\varepsilon_{p_1 p_2}) = -1$, then

$$\mathfrak{A}^{2^n} \mathfrak{P}^{2^{m-1}} \sim 2_{F_1}^{2^{n-1}} I^{2^{m-2}} \sim \mathcal{P}_1 \sim \mathcal{H}_1.$$

- If $\left(\frac{p_1}{p_2}\right) = 1$ and $N(\varepsilon_{p_1 p_2}) = 1$, then $2_{F_1}^{2^n} \sim 1$, thus

$$\mathfrak{P}^{2^{m-1}} \sim I^{2^{m-2}} \sim \mathcal{P}_1 \sim \mathcal{H}_1.$$

Finally, from Lemma 6, we deduce the following remark

Remark 1. (1) Assume $q = 1$, so

- (i) If $\left(\frac{p_1}{p_2}\right) = -1$ or $\left(\frac{p_1}{p_2}\right) = 1$ and $\left(\frac{p_1}{p_2}\right)_4 \left(\frac{p_2}{p_1}\right)_4 = -1$, then $n = 1$ and $m \geq 3$, thus $\mathbf{Cl}_2(\mathbb{K}_3) \simeq (4, 2^m)$.
- (ii) If $\left(\frac{p_1}{p_2}\right) = 1$ and $\left(\frac{p_1}{p_2}\right)_4 \left(\frac{p_2}{p_1}\right)_4 = 1$, then $m = 2$ and $n \geq 2$, hence $\mathbf{Cl}_2(\mathbb{K}_3) \simeq (4, 2^{n+1})$.

- (2) If $q = 2$, then $\mathbf{Cl}_2(\mathbb{K}_3) \simeq (4, 2^{n+2})$.

This completes the proof of the second assertion.

- (3) For the proof of the third assertion see [4].

- (4) **Computation of $\text{Gal}(\mathbb{k}_2^{(2)}/\mathbb{k})$.**

Put $L = \mathbb{k}_2^{(2)}$, the Hilbert 2-class field of \mathbb{k} . Let $\left(\frac{L/\mathbb{K}_3}{P}\right)$ denote the Artin symbol for the normal extension L/\mathbb{K}_3 ; hence it is clear that $\sigma = \left(\frac{L/\mathbb{K}_3}{\mathfrak{P}}\right)$ and $\tau = \left(\frac{L/\mathbb{K}_3}{\mathfrak{A}}\right)$ generate the abelian subgroup $\text{Gal}(L/\mathbb{K}_3)$ of $G = \text{Gal}(L/\mathbb{k})$. If we

put also $\rho = \left(\frac{L/\mathbb{k}}{\mathcal{H}_1}\right)$, then ρ restricts to the nontrivial automorphism of \mathbb{K}_3/\mathbb{k} , since \mathcal{H}_1 is not norm from \mathbb{K}_3/\mathbb{k} ; from which we deduce that

$$G = \text{Gal}(L/\mathbb{k}) = \langle \rho, \tau, \sigma \rangle.$$

$$\text{Note that } |G| = 2|\text{Gal}(L/\mathbb{K}_3)| = \begin{cases} 2^{n+m+2} & \text{if } q = 1, \\ 2^{n+m+3} & \text{if } q = 2. \end{cases}$$

To continue, let us prove the following result.

Lemma 8. *In $\mathbf{Cl}_2(\mathbb{k})$, we have $N_{\mathbb{K}_3/\mathbb{k}}(\mathfrak{P}) \sim \mathcal{H}_1\mathcal{H}_2$.*

Proof. We choose a prime ideal \mathfrak{A} in \mathbb{K}_3 such that $[\mathfrak{A}] = [\mathfrak{P}]$, this is always possible by Chebotarev's theorem, hence $\mathcal{R}_{\mathbb{k}} \sim N_{\mathbb{K}_3/\mathbb{k}}(\mathfrak{A})$ and $\mathcal{R}_{F_1} \sim N_{\mathbb{K}_3/F_1}(\mathfrak{A})$ are prime ideals in \mathbb{k} and F_1 respectively, thus $\mathcal{R}_{F_1} \sim N_{\mathbb{K}_3/F_1}(\mathfrak{A}) \sim N_{\mathbb{K}_3/F_1}(\mathfrak{P}) \sim I$. As the extension F_1/k_1 is ramified and $\mathbf{Cl}_2(k_1)$ is generated by z_1 , we infer that the prime ideal $\mathcal{R}_{k_1} \sim N_{F_1/k_1}(\mathcal{R}_{F_1}) \sim N_{F_1/k_1}(I) \sim z_1^{2^i}$ with some integer i . This implies that $2^{2^i}r = \pm(x^2 - p_1p_2y^2)$, which in turn shows that $(\frac{r}{p_1}) = 1$.

We know that $N_{\mathbb{K}_3/\mathbb{k}}(\mathbf{Cl}_2(\mathbb{K}_3)) = \langle [\mathcal{H}_0], [\mathcal{H}_1\mathcal{H}_2] \rangle$. So if $N_{\mathbb{K}_3/\mathbb{k}}(\mathfrak{P}) \sim \mathcal{H}_0$, then $\mathcal{R}_{\mathbb{k}} \sim \mathcal{H}_0$ (equivalence in $\mathbf{Cl}_2(\mathbb{k})$); hence the prime ideal $\mathfrak{r} = N_{\mathbb{k}/k_0}(\mathcal{R}_{\mathbb{k}})$ of k_0 is equivalent, in $\mathbf{Cl}_2(k_0)$, to $\tilde{z} \sim P_1P_2$. Therefore the equivalence $\mathfrak{r} \sim \tilde{z}$ yields that $2r = \pm(x^2 - 2p_1p_2y^2)$, where x, y are in \mathbb{Z} ; which shows that $(\frac{2r}{p_1}) = 1$, this leads to the contradiction $(\frac{r}{p_1}) = -1$, since $(\frac{2}{p_1}) = -1$. We get the same contradiction if we suppose that $N_{\mathbb{K}_3/\mathbb{k}}(\mathfrak{P}) \sim \mathcal{H}_0\mathcal{H}_1\mathcal{H}_2$. Finally the equivalence $N_{\mathbb{K}_3/\mathbb{k}}(\mathfrak{P}) \sim 1$ can not occur since the order of σ is strictly greater than 1. \square

Therefore the following relations hold:

- $[\tau, \sigma] = 1$.
 - $\rho^2 = \left(\frac{L/\mathbb{k}}{\mathcal{H}_1^2}\right) = \left(\frac{L/\mathbb{k}}{N_{\mathbb{K}_3/\mathbb{k}}(\mathcal{H}_1)}\right) = \left(\frac{L/\mathbb{K}_3}{\mathcal{H}_1}\right)$, so $\rho^4 = 1$.
 - $\tau\rho^{-1}\tau\rho = \left(\frac{L/\mathbb{K}_3}{\mathfrak{A}^{1+\rho}}\right) = 1$, since $\mathfrak{A}^{1+\rho} = N_{\mathbb{K}_3/\mathbb{k}}(\mathfrak{A}) \sim \mathcal{H}_0 \sim 1$, thus $[\tau, \rho] = \tau^{-1}\rho^{-1}\tau\rho = \tau^{-2}$ and $[\rho, \tau] = \tau^2$.
 - $\sigma\rho^{-1}\sigma\rho = \left(\frac{L/\mathbb{K}_3}{\mathfrak{P}^{1+\rho}}\right) = \begin{cases} 1, & \text{if } q = 1, \\ \sigma^{2^m}, & \text{if } q = 2, \end{cases}$ since, in $\mathbf{Cl}_2(\mathbb{K}_3)$, we have
- $$\mathfrak{P}^{1+\rho} = N_{\mathbb{K}_3/\mathbb{k}}(\mathfrak{P}) \sim \mathcal{H}_1\mathcal{H}_2 \sim \begin{cases} 1, & \text{if } q = 1, \\ \mathfrak{P}^{2^m}, & \text{if } q = 2, \end{cases}$$
- therefore $[\sigma, \rho] = \begin{cases} \sigma^{-2}, & \text{if } q = 1, \\ \sigma^{2^m-2} = \sigma^2, & \text{if } q = 2, \end{cases}$ since in this case $m = 2$.

- Suppose $q = 1$, so
 - If $\left(\frac{p_1}{p_2}\right) = -1$ or $\left(\frac{p_1}{p_2}\right) = 1$ and $N(\varepsilon_{p_1 p_2}) = -1$, then $\rho^4 = \sigma^{2^m} = \tau^{2^{n+1}} = 1$ and $\rho^2 = \sigma^{2^{m-1}} \tau^{2^n}$, since $\mathfrak{A}^{2^{n+1}} \sim \mathfrak{P}^{2^m} \sim 1$ and $\mathcal{H}_1 \sim \mathfrak{A}^{2^n} \mathfrak{P}^{2^{m-1}}$.
 - If $\left(\frac{p_1}{p_2}\right) = 1$ and $N(\varepsilon_{p_1 p_2}) = 1$, then $\rho^4 = \sigma^{2^m} = \tau^{2^{n+1}} = 1$ and $\rho^2 = \sigma^{2^{m-1}}$ since $\mathcal{H}_1 \sim \mathfrak{P}^{2^{m-1}}$.
- Suppose $q = 2$, so necessarily $\left(\frac{p_1}{p_2}\right) = -1$ or $\left(\frac{p_1}{p_2}\right) = 1$ and $N(\varepsilon_{p_1 p_2}) = -1$, then

$$\begin{cases} \rho^4 = \sigma^{2^{m+1}} = \tau^{2^{n+2}} = 1, \\ \sigma^{2^m} = \tau^{2^{n+1}} \text{ and } \rho^2 = \sigma^{2^{m-1}} \tau^{2^n}, \end{cases}$$
 since $\mathfrak{A}^{2^{n+2}} \sim \mathfrak{P}^{2^{m+1}} \sim 1$, $\mathfrak{A}^{2^{n+1}} \sim \mathfrak{P}^{2^m}$ and $\mathcal{H}_1 \sim \mathfrak{A}^{2^n} \mathfrak{P}^{2^{m-1}}$.

(5) As $[\tau, \sigma] = 1$, $[\rho, \sigma] = \sigma^2$ or σ^{-2} and $[\rho, \tau] = \tau^2$, then the derived group of G is $G' = \langle \sigma^2, \tau^2 \rangle$, therefore

$$\mathbf{Cl}_2(\mathbb{K}_2^{(1)}) \simeq \begin{cases} (2^{m-1}, 2^n) & \text{if } q = 1, \\ (2^{\min(m, n+1)-1}, 2^{\max(m+1, n+2)-1}) = (2, 2^{n+1}) & \text{if } q = 2. \end{cases}$$

(6) Finally, we compute the coclass of G .

Let G be the group defined above. Then the lower central series of G is defined inductively by $\gamma_1(G) = G$ and $\gamma_{i+1}(G) = [\gamma_i(G), G]$, that is the subgroup of G generated by the set $\{[a, b] = a^{-1}b^{-1}ab/a \in \gamma_i(G), b \in G\}$, so the coclass of G is defined to be $cc(G) = h - c$, where $|G| = 2^h$ and $c = c(G)$ is the nilpotency class of G , that is the smallest integer satisfying $\gamma_{c+1}(G) = 1$. We easily get

$$\gamma_1(G) = G.$$

$$\gamma_2(G) = G' = \langle \sigma^2, \tau^2 \rangle.$$

$$\gamma_3(G) = [G', G] = \langle \sigma^4, \tau^4 \rangle.$$

Then Proposition 3(6) (see below) yields that $\gamma_{j+1}(G) = [\gamma_j(G), G] = \langle \sigma^{2^j}, \tau^{2^j} \rangle$.

Suppose $q = 1$, then if we put $v = \max(n, m-1)$, we get

$$\gamma_{v+2}(G) = \langle \sigma^{2^{v+1}}, \tau^{2^{v+1}} \rangle = \langle 1 \rangle \text{ and } \gamma_{v+1}(G) = \langle \sigma^{2^v}, \tau^{2^v} \rangle \neq \langle 1 \rangle. \text{ As, in this case, } |G| = 2^{n+m+2}, \text{ so}$$

$$c(G) = v + 1 \text{ and } cc(G) = n + m + 1 - v = 3,$$

in fact, from Lemma 6, we have $m \geq 3$ and $n = 1$ or $m = 2$ and $n \geq 2$, so the first case implies that $v = m - 1$ and $cc(G) = n + m + 1 - v = 3$, whereas the second one yields that $v = n$ and $cc(G) = n + m + 1 - v = 3$.

Suppose $q = 2$, then if we put $v = \max(n+1, m)$, we get

$$\gamma_{v+2}(G) = \langle \sigma^{2^{v+1}}, \tau^{2^{v+1}} \rangle = \langle 1 \rangle \text{ and } \gamma_{v+1}(G) = \langle \sigma^{2^v}, \tau^{2^v} \rangle \neq \langle 1 \rangle. \text{ As, in this case,}$$

$|G| = 2^{n+m+3}$, so

$$c(G) = v + 1 \text{ and } cc(G) = n + m + 2 - v = 3,$$

since in this case, from Lemma 6, $m = 2$ and $n \geq 1$, thus $v = n + 1$.

4.2. Proof of Theorem 3. For this we need the following results which are easy to check.

Proposition 3. *Let $G = \langle \sigma, \tau, \rho \rangle$ denote the group defined above, then*

- (1) $\rho^{-1}\sigma\rho = \begin{cases} \sigma^{-1} & \text{if } q = 1, \\ \sigma^3 & \text{if } q = 2. \end{cases}$
- (2) $\rho^{-1}\tau\rho = \tau^{-1}$.
- (3) $[\rho^2, \sigma] = [\rho^2, \tau] = 1$.
- (4) $(\tau\rho)^2 = \rho^2$.
- (5) $(\sigma\tau\rho)^2 = (\sigma\rho)^2 = \begin{cases} \rho^2 & \text{if } q = 1, \\ \rho^2\sigma^4 & \text{if } q = 2. \end{cases}$
- (6) For all $r \in \mathbb{N}$, $[\rho, \tau^{2^r}] = \tau^{2^{r+1}}$ and $[\rho, \sigma^{2^r}] = \begin{cases} \sigma^{2^{r+1}} & \text{if } q = 1, \\ \sigma^{-2^{r+1}} & \text{if } q = 2. \end{cases}$

The proof of Theorem 3 consists of 3 parts. In the first part, we will compute $N_{\mathbb{K}_j/\mathbb{K}}(\mathbf{Cl}_2(\mathbb{K}_j))$, for all $1 \leq j \leq 7$. In the second one, we will determine the capitulation kernels $\kappa_{\mathbb{K}_j}$ and the types of $\mathbf{Cl}_2(\mathbb{K}_4)$ and in the third one, we will determine the capitulation kernels $\kappa_{\mathbb{L}_j}$ and the types of $\mathbf{Cl}_2(\mathbb{L}_4)$. It should be noted that if $\left(\frac{p_1}{p_2}\right) = -1$, then Propositions 1, 2 imply that

$$\begin{cases} q = 1 \Leftrightarrow \left(\frac{\pi_1}{\pi_3}\right) = \left(\frac{1+i}{\pi_1}\right) \left(\frac{1+i}{\pi_3}\right) \\ q = 2 \Leftrightarrow \left(\frac{\pi_1}{\pi_3}\right) = - \left(\frac{1+i}{\pi_1}\right) \left(\frac{1+i}{\pi_3}\right) \end{cases}$$

4.2.1. Norm class groups. Let us compute $N_j = N_{\mathbb{K}_j/\mathbb{K}}(\mathbf{Cl}_2(\mathbb{K}_j))$, the results are summarized in the following table. Note that the left hand sides refer to the case $\left(\frac{\pi_1}{\pi_3}\right) = -1$, while the right ones refer to the case $\left(\frac{\pi_1}{\pi_3}\right) = 1$. Put $B = \left(\frac{1+i}{\pi_1}\right) \left(\frac{1+i}{\pi_3}\right)$.

Table 1: Norm class groups

\mathbb{K}_j	N_j for $\left(\frac{p_1}{p_2}\right) = 1$	N_j for $\left(\frac{p_1}{p_2}\right) = -1$
\mathbb{K}_1	$\langle [\mathcal{H}_0\mathcal{H}_1], [\mathcal{H}_0\mathcal{H}_2] \rangle$	$\langle [\mathcal{H}_1], [\mathcal{H}_2] \rangle$
\mathbb{K}_2	$\langle [\mathcal{H}_1], [\mathcal{H}_2] \rangle$	$\langle [\mathcal{H}_0\mathcal{H}_1], [\mathcal{H}_0\mathcal{H}_2] \rangle$
\mathbb{K}_3	$\langle [\mathcal{H}_0], [\mathcal{H}_1\mathcal{H}_2] \rangle$	$\langle [\mathcal{H}_0], [\mathcal{H}_1\mathcal{H}_2] \rangle$

\mathbb{K}_j	N_j for $\left(\frac{p_1}{p_2}\right) = 1$	N_j for $\left(\frac{p_1}{p_2}\right) = -1$
\mathbb{K}_4	$\langle [\mathcal{H}_0], [\mathcal{H}_2] \rangle, \langle [\mathcal{H}_0], [\mathcal{H}_1] \rangle$ if $B = 1$ $\langle [\mathcal{H}_2], [\mathcal{H}_0\mathcal{H}_1] \rangle, \langle [\mathcal{H}_1], [\mathcal{H}_0\mathcal{H}_2] \rangle$ if $B = -1$	$\langle [\mathcal{H}_1], [\mathcal{H}_0\mathcal{H}_2] \rangle \langle [\mathcal{H}_0], [\mathcal{H}_2] \rangle$ if $q = 1$ $\langle [\mathcal{H}_0], [\mathcal{H}_1] \rangle \langle [\mathcal{H}_2], [\mathcal{H}_0\mathcal{H}_1] \rangle$ if $q = 2$
\mathbb{K}_5	$\langle [\mathcal{H}_2], [\mathcal{H}_0\mathcal{H}_1] \rangle, \langle [\mathcal{H}_1], [\mathcal{H}_0\mathcal{H}_2] \rangle$ if $B = 1$ $\langle [\mathcal{H}_0], [\mathcal{H}_2] \rangle, \langle [\mathcal{H}_0], [\mathcal{H}_1] \rangle$ if $B = -1$	$\langle [\mathcal{H}_0], [\mathcal{H}_2] \rangle \langle [\mathcal{H}_1], [\mathcal{H}_0\mathcal{H}_2] \rangle$ if $q = 1$ $\langle [\mathcal{H}_2], [\mathcal{H}_0\mathcal{H}_1] \rangle \langle [\mathcal{H}_0], [\mathcal{H}_1] \rangle$ if $q = 2$
\mathbb{K}_6	$\langle [\mathcal{H}_1], [\mathcal{H}_0\mathcal{H}_2] \rangle, \langle [\mathcal{H}_2], [\mathcal{H}_0\mathcal{H}_1] \rangle$ if $B = 1$ $\langle [\mathcal{H}_0], [\mathcal{H}_1] \rangle, \langle [\mathcal{H}_0], [\mathcal{H}_2] \rangle$ if $B = -1$	$\langle [\mathcal{H}_0], [\mathcal{H}_1] \rangle \langle [\mathcal{H}_2], [\mathcal{H}_0\mathcal{H}_1] \rangle$ if $q = 1$ $\langle [\mathcal{H}_1], [\mathcal{H}_0\mathcal{H}_2] \rangle \langle [\mathcal{H}_0], [\mathcal{H}_2] \rangle$ if $q = 2$
\mathbb{K}_7	$\langle [\mathcal{H}_0], [\mathcal{H}_1] \rangle, \langle [\mathcal{H}_0], [\mathcal{H}_2] \rangle$ if $B = 1$ $\langle [\mathcal{H}_1], [\mathcal{H}_0\mathcal{H}_2] \rangle, \langle [\mathcal{H}_2], [\mathcal{H}_0\mathcal{H}_1] \rangle$ if $B = -1$	$\langle [\mathcal{H}_2], [\mathcal{H}_0\mathcal{H}_1] \rangle \langle [\mathcal{H}_0], [\mathcal{H}_1] \rangle$ if $q = 1$ $\langle [\mathcal{H}_0], [\mathcal{H}_2] \rangle \langle [\mathcal{H}_1], [\mathcal{H}_0\mathcal{H}_2] \rangle$ if $q = 2$

To check the table entries we use Lemmas 2, 7, Propositions 1, 2 and the following results which are easy to prove.

Lemma 9. *Let $p_1 \equiv p_2 \equiv 1 \pmod{4}$ be primes. Put $p_1 = \pi_1\pi_2$ and $p_2 = \pi_3\pi_4$, where $\pi_j \in \mathbb{Z}[i]$, then*

- (i) $\left(\frac{\pi_1}{\pi_2}\right) = \left(\frac{\pi_3}{\pi_4}\right) = \begin{cases} 1 & \text{if } p_1 \equiv p_2 \equiv 1 \pmod{8}, \\ -1 & \text{if } p_1 \equiv p_2 \equiv 5 \pmod{8} \end{cases}$
- (ii) If $\left(\frac{p_1}{p_2}\right) = 1$, then $\left(\frac{\pi_1}{\pi_3}\right) = \left(\frac{\pi_2}{\pi_3}\right) = \left(\frac{\pi_1}{\pi_4}\right) = \left(\frac{\pi_2}{\pi_4}\right)$.
- (iii) If $\left(\frac{p_1}{p_2}\right) = -1$, then $\left(\frac{\pi_1}{\pi_3}\right) = \left(\frac{\pi_2}{\pi_4}\right) = -\left(\frac{\pi_2}{\pi_3}\right) = -\left(\frac{\pi_1}{\pi_4}\right)$.
- (iv) If $\left(\frac{2}{p_1}\right) = 1$, then $\left(\frac{1+i}{\pi_1}\right) = \left(\frac{1+i}{\pi_2}\right)$.
- (v) If $\left(\frac{2}{p_1}\right) = -1$, then $\left(\frac{1+i}{\pi_1}\right) = -\left(\frac{1+i}{\pi_2}\right)$.

Compute N_j in a few cases. Keeping in mind that $\mathcal{H}_0, \mathcal{H}_1$ and \mathcal{H}_2 are unramified in \mathbb{K}_j/\mathbb{k} .

Take first $\mathbb{K}_1 = \mathbb{k}(\sqrt{p_1}) = \mathbb{k}(\sqrt{2p_2}) = \mathbb{Q}(\sqrt{p_1}, \sqrt{2p_2}, i)$. As $N_1 = \{[\mathcal{H}] \in \mathbf{Cl}_2(\mathbb{k}) / \left(\frac{\alpha}{\mathcal{H}}\right) = 1\}$, so for $j \in \{1, 2\}$ we get

$$\begin{aligned}
\left(\frac{\mathbb{k}(\sqrt{2p_2})/\mathbb{k}}{\mathcal{H}_j}\right) &= \left(\frac{\mathbb{k}(\sqrt{2p_2})/\mathbb{k}}{\mathcal{H}_j}\right) (\sqrt{2p_2})(\sqrt{2p_2})^{-1} \\
&= \left(\frac{2p_2}{\mathcal{H}_j}\right) \\
&= \left(\frac{2p_2}{p_1}\right) \\
&= \left(\frac{2}{p_1}\right) \left(\frac{p_1}{p_2}\right).
\end{aligned}$$

Thus

- If $\left(\frac{p_1}{p_2}\right) = -1$, then $[\mathcal{H}_j] \in N_1$, hence $N_1 = \langle [\mathcal{H}_1], [\mathcal{H}_2] \rangle$.
- If $\left(\frac{p_1}{p_2}\right) = 1$, then $[\mathcal{H}_j] \notin N_1$, hence $[\mathcal{H}_1\mathcal{H}_2] \in N_1$. Moreover, since $\left(\frac{2}{p_1}\right) = -1$, then $\mathcal{H}_0 \notin N_1$; from which we deduce that $[\mathcal{H}_0\mathcal{H}_1]$ and $[\mathcal{H}_0\mathcal{H}_2]$ are in N_1 , therefore

$$N_1 = \langle [\mathcal{H}_0\mathcal{H}_1], [\mathcal{H}_0\mathcal{H}_2] \rangle.$$

Take an other example, $\mathbb{K}_4 = \mathbb{k}(\sqrt{\pi_1\pi_3}) = \mathbb{k}(\sqrt{2\pi_2\pi_4})$. First prove that $\left(\frac{\pi_1\pi_3}{\mathcal{H}_0}\right) = \left(\frac{1+i}{\pi_1}\right) \left(\frac{1+i}{\pi_3}\right)$. As $1+i$ is unramified in both of $\mathbb{Q}(\sqrt{\pi_1\pi_3})/\mathbb{Q}(i)$ and $\mathbb{k}/\mathbb{Q}(i)$, so according to [11, Proposition 4.2, p.112] and Hilbert symbol properties we get

$$\begin{aligned} \left(\frac{\pi_1\pi_3}{\mathcal{H}_0}\right) &= \left(\frac{\pi_1\pi_3}{1+i}\right) = \left(\frac{\pi_1\pi_3}{1+i}\right)^{v_{1+i}(1+i)} \\ &= \left(\frac{1+i, \pi_1\pi_3}{1+i}\right) \\ &= \left(\frac{1+i, \pi_1}{1+i}\right) \left(\frac{1+i, \pi_3}{1+i}\right). \end{aligned}$$

On the other hand, the product formula implies, for $j \in \{1, 2\}$, that

$$\left(\frac{1+i, \pi_j}{1+i}\right) \left(\frac{1+i, \pi_j}{\pi_j}\right) \prod_{\mathcal{P} \neq \pi_j, \mathcal{P} \neq 1+i} \left(\frac{1+i, \pi_j}{\mathcal{P}}\right) = 1;$$

as \mathcal{P} does not divide π_j and $1+i$, so $\left(\frac{1+i, \pi_j}{\mathcal{P}}\right) = 1$, which yields that

$$\left(\frac{1+i, \pi_j}{1+i}\right) \left(\frac{1+i, \pi_j}{\pi_j}\right) = 1, \text{ hence } \left(\frac{1+i, \pi_j}{1+i}\right) = \left(\frac{1+i, \pi_j}{\pi_j}\right) = \left(\frac{1+i}{\pi_j}\right).$$

This implies the result.

Compute now N_4 .

$$\text{We have } \begin{cases} \left(\frac{\pi_1\pi_3}{\mathcal{H}_2}\right) = \left(\frac{\pi_1\pi_3}{\pi_2}\right) = \left(\frac{\pi_1}{\pi_2}\right) \left(\frac{\pi_3}{\pi_2}\right) = -\left(\frac{\pi_2}{\pi_3}\right), \\ \left(\frac{2\pi_2\pi_4}{\mathcal{H}_1}\right) = \left(\frac{2\pi_2\pi_4}{\pi_1}\right) = \left(\frac{2}{p_1}\right) \left(\frac{\pi_1}{\pi_2}\right) \left(\frac{\pi_4}{\pi_1}\right) = \left(\frac{\pi_1}{\pi_4}\right), \\ \left(\frac{\pi_1\pi_3}{\mathcal{H}_0}\right) = \left(\frac{1+i}{\pi_1}\right) \left(\frac{1+i}{\pi_3}\right). \end{cases}$$

Assume that $\left(\frac{p_1}{p_2}\right) = 1$. So

- If $\left(\frac{p_1}{p_2}\right)_4 \left(\frac{p_2}{p_1}\right)_4 = \left(\frac{\pi_1}{\pi_3}\right) = -1$, then $\mathcal{H}_2 \in N_4$ and $\mathcal{H}_1 \notin N_4$;
- thus $\begin{cases} \text{If } \left(\frac{1+i}{\pi_1}\right) \left(\frac{1+i}{\pi_3}\right) = 1, \text{ then } N_4 = \langle [\mathcal{H}_0], [\mathcal{H}_2] \rangle, \\ \text{If } \left(\frac{1+i}{\pi_1}\right) \left(\frac{1+i}{\pi_3}\right) = -1, \text{ then } N_4 = \langle [\mathcal{H}_2], [\mathcal{H}_0\mathcal{H}_1] \rangle. \end{cases}$
- If $\left(\frac{p_1}{p_2}\right)_4 \left(\frac{p_2}{p_1}\right)_4 = \left(\frac{\pi_1}{\pi_3}\right) = 1$, then $\mathcal{H}_2 \notin N_4$ and $\mathcal{H}_1 \in N_4$;

hence $\begin{cases} \text{If } \left(\frac{1+i}{\pi_1}\right)\left(\frac{1+i}{\pi_3}\right) = 1, \text{ then } N_4 = \langle [\mathcal{H}_0], [\mathcal{H}_1] \rangle, \\ \text{If } \left(\frac{1+i}{\pi_1}\right)\left(\frac{1+i}{\pi_3}\right) = -1, \text{ then } N_4 = \langle [\mathcal{H}_1], [\mathcal{H}_0\mathcal{H}_2] \rangle. \end{cases}$

Assume that $\left(\frac{p_1}{p_2}\right) = -1$. So

- If $q = 1$, then $\left(\frac{\pi_1}{\pi_3}\right) = \left(\frac{1+i}{\pi_1}\right)\left(\frac{1+i}{\pi_3}\right)$, hence

$$\begin{cases} \text{If } \left(\frac{\pi_1}{\pi_3}\right) = 1, \text{ then } N_4 = \langle [\mathcal{H}_0], [\mathcal{H}_2] \rangle, \\ \text{If } \left(\frac{\pi_1}{\pi_3}\right) = -1, \text{ then } N_4 = \langle [\mathcal{H}_1], [\mathcal{H}_0\mathcal{H}_2] \rangle. \end{cases}$$
- If $q = 2$, then $\left(\frac{\pi_1}{\pi_3}\right) = -\left(\frac{1+i}{\pi_1}\right)\left(\frac{1+i}{\pi_3}\right)$, hence

$$\begin{cases} \text{If } \left(\frac{\pi_1}{\pi_3}\right) = 1, \text{ then } N_4 = \langle [\mathcal{H}_2], [\mathcal{H}_0\mathcal{H}_1] \rangle, \\ \text{If } \left(\frac{\pi_1}{\pi_3}\right) = -1, \text{ then } N_4 = \langle [\mathcal{H}_0], [\mathcal{H}_1] \rangle. \end{cases}$$

Proceeding similarly, we check the other table inputs.

4.2.2. Capitulation kernels $\kappa_{\mathbb{K}_j}$ and $\text{Gal}(\mathbb{K}_2^{(2)}/\mathbb{K}_j)$. Let us compute the Galois groups $G_j = \text{Gal}(\mathbb{K}_2^{(2)}/\mathbb{K}_j)$, the capitulation kernels $\kappa_{\mathbb{K}_j}$, $\kappa_{\mathbb{K}_j} \cap N_j$ and the types of $\mathbf{Cl}_2(\mathbb{K}_j)$. The results are summarized in the following tables. Note that the left hand sides refer to the case $\left(\frac{\pi_1}{\pi_3}\right) = -1$, while the right ones refer to the case $\left(\frac{\pi_1}{\pi_3}\right) = 1$. Put $B = \left(\frac{1+i}{\pi_1}\right)\left(\frac{1+i}{\pi_3}\right)$, $a = \min(m, n+1)$ and $b = \max(m+1, n+2)$.

Table 2: $\kappa_{\mathbb{K}_j}$ for the case $\left(\frac{p_1}{p_2}\right) = 1$.

\mathbb{K}_j		G_j	$\kappa_{\mathbb{K}_j}$	$\kappa_{\mathbb{K}_j} \cap N_j$	$\mathbf{Cl}_2(\mathbb{K}_j)$
\mathbb{K}_1		$\langle \sigma, \tau\rho, \tau^2 \rangle$	$\langle [\mathcal{H}_1], [\mathcal{H}_2] \rangle$	$\langle [\mathcal{H}_1\mathcal{H}_2] \rangle$	$(2, 2, 2)$
\mathbb{K}_2		$\langle \sigma, \rho, \tau^2 \rangle$	$\langle [\mathcal{H}_0\mathcal{H}_1], [\mathcal{H}_0\mathcal{H}_2] \rangle$	$\langle [\mathcal{H}_1\mathcal{H}_2] \rangle$	$(2, 2, 2)$
\mathbb{K}_3	$q = 1$ $q = 2$	$\langle \tau, \sigma \rangle$	$\langle [\mathcal{H}_0], [\mathcal{H}_1\mathcal{H}_2] \rangle$ $\langle [\mathcal{H}_0] \rangle$	N_3 $\langle [\mathcal{H}_0] \rangle$	$(2^m, 2^{n+1})$ $(2^a, 2^b)$
\mathbb{K}_4	$B = 1$ $B = -1$	$\langle \tau, \rho\sigma, \sigma^2 \rangle \langle \tau, \rho \rangle$ $\langle \sigma\tau, \tau\rho, \sigma^2 \rangle \langle \sigma\tau, \rho \rangle$	$\langle [\mathcal{H}_0], [\mathcal{H}_1] \rangle$ $\langle [\mathcal{H}_1], [\mathcal{H}_0\mathcal{H}_2] \rangle$	$\langle [\mathcal{H}_0] \rangle$ $\langle [\mathcal{H}_0\mathcal{H}_1\mathcal{H}_2] \rangle$ N_4	$(2, 2, 2) \ (2, 4)$
\mathbb{K}_5	$B = 1$ $B = -1$	$\langle \sigma\tau, \tau\rho, \sigma^2 \rangle \langle \sigma\tau, \rho \rangle$ $\langle \tau, \rho\sigma, \sigma^2 \rangle \langle \tau, \rho \rangle$	$\langle [\mathcal{H}_1], [\mathcal{H}_0\mathcal{H}_2] \rangle$ $\langle [\mathcal{H}_0], [\mathcal{H}_1] \rangle$	$\langle [\mathcal{H}_0\mathcal{H}_1\mathcal{H}_2] \rangle$ $\langle [\mathcal{H}_0] \rangle$ N_5	$(2, 2, 2) \ (2, 4)$
\mathbb{K}_6	$B = 1$ $B = -1$	$\langle \sigma\tau, \rho, \sigma^2 \rangle \langle \sigma\tau, \tau\rho \rangle$ $\langle \tau, \rho, \sigma^2 \rangle \langle \tau, \rho\sigma \rangle$	$\langle [\mathcal{H}_2], [\mathcal{H}_0\mathcal{H}_1] \rangle$ $\langle [\mathcal{H}_0], [\mathcal{H}_2] \rangle$	$\langle [\mathcal{H}_0\mathcal{H}_1\mathcal{H}_2] \rangle$ $\langle [\mathcal{H}_0] \rangle$ N_6	$(2, 2, 2) \ (2, 4)$
\mathbb{K}_7	$B = 1$ $B = -1$	$\langle \tau, \rho, \sigma^2 \rangle \langle \tau, \rho\sigma \rangle$ $\langle \sigma\tau, \rho, \sigma^2 \rangle \langle \sigma\tau, \tau\rho \rangle$	$\langle [\mathcal{H}_0], [\mathcal{H}_2] \rangle$ $\langle [\mathcal{H}_2], [\mathcal{H}_0\mathcal{H}_1] \rangle$	$\langle [\mathcal{H}_0] \rangle$ $\langle [\mathcal{H}_0\mathcal{H}_1\mathcal{H}_2] \rangle$ N_7	$(2, 2, 2) \ (2, 4)$

Table 3: $\kappa_{\mathbb{K}_j}$ for the case $\left(\frac{p_1}{p_2}\right) = -1$.

\mathbb{K}_j	G_j	$\kappa_{\mathbb{K}_j}$	$\kappa_{\mathbb{K}_j} \cap N_j$	$\mathbf{Cl}_2(\mathbb{K}_j)$
\mathbb{K}_1	$\langle \sigma, \rho \rangle$	$\langle [\mathcal{H}_1], [\mathcal{H}_2] \rangle$	N_1	$(2, 4)$
\mathbb{K}_2	$\langle \sigma, \tau \rho \rangle$	$\langle [\mathcal{H}_0 \mathcal{H}_1], [\mathcal{H}_0 \mathcal{H}_2] \rangle$	N_2	$(2, 4)$
\mathbb{K}_3	$q = 1$	$\langle [\mathcal{H}_0], [\mathcal{H}_1 \mathcal{H}_2] \rangle$	N_3	$(4, 2^m)$
	$q = 2$	$\langle [\mathcal{H}_0] \rangle$	$\langle [\mathcal{H}_0] \rangle$	$(4, 2^{m+1})$
\mathbb{K}_4	$q = 1$	$\langle \rho, \tau \sigma \rangle \langle \tau, \rho \sigma, \sigma^2 \rangle$	N_4	$\langle [\mathcal{H}_0] \rangle$
	$q = 2$	$\langle \tau, \rho \rangle \langle \sigma \tau, \tau \rho, \sigma^2 \rangle$		
\mathbb{K}_5	$q = 1$	$\langle \tau, \rho \sigma, \sigma^2 \rangle \langle \rho, \sigma \tau \rangle$	N_5	$\langle [\mathcal{H}_0] \rangle$
	$q = 2$	$\langle \sigma \tau, \tau \rho, \sigma^2 \rangle \langle \tau, \rho \rangle$		
\mathbb{K}_6	$q = 1$	$\langle \tau, \rho, \sigma^2 \rangle \langle \sigma \tau, \tau \rho \rangle$	N_6	$\langle [\mathcal{H}_0] \rangle$
	$q = 2$	$\langle \rho, \sigma \tau, \sigma^2 \rangle \langle \tau, \rho \sigma \rangle$		
\mathbb{K}_7	$q = 1$	$\langle \sigma \tau, \tau \rho \rangle \langle \rho, \tau, \sigma^2 \rangle$	N_7	$\langle [\mathcal{H}_0] \rangle$
	$q = 2$	$\langle \tau, \rho \sigma \rangle \langle \rho, \sigma \tau, \sigma^2 \rangle$		

Before proving these results, note that, from Tables 1, 2 and 3 we get the following remark:

Remark 2. Put $B = \left(\frac{1+i}{\pi_1}\right)\left(\frac{1+i}{\pi_3}\right)$ and $\pi = \left(\frac{\pi_1}{\pi_3}\right)$.

- (1) $\begin{cases} \kappa_{\mathbb{K}_1} = N_2, & \kappa_{\mathbb{K}_2} = N_1 \text{ if } \left(\frac{p_1}{p_2}\right) = 1, \\ \kappa_{\mathbb{K}_1} = N_1, & \kappa_{\mathbb{K}_2} = N_2 \text{ if } \left(\frac{p_1}{p_2}\right) = -1. \end{cases}$
- (2) Assume that $\left(\frac{p_1}{p_2}\right) = 1$, so
 - If $\pi = 1$, then $\kappa_{\mathbb{K}_4} = N_4$, $\kappa_{\mathbb{K}_5} = N_5$, $\kappa_{\mathbb{K}_6} = N_6$ and $\kappa_{\mathbb{K}_7} = N_7$.
 - Else $\kappa_{\mathbb{K}_4} = N_7$, $\kappa_{\mathbb{K}_5} = N_6$, $\kappa_{\mathbb{K}_6} = N_5$ and $\kappa_{\mathbb{K}_7} = N_4$.

- (3) Assume that $\left(\frac{p_1}{p_2}\right) = -1$ and $q = 1$, so

$$\begin{aligned} \kappa_{\mathbb{K}_4} &= \begin{cases} N_4 & \text{if } \pi = -1, \\ N_7 & \text{if } \pi = 1. \end{cases} & \kappa_{\mathbb{K}_5} &= \begin{cases} N_6 & \text{if } \pi = -1, \\ N_5 & \text{if } \pi = 1. \end{cases} \\ \kappa_{\mathbb{K}_6} &= \begin{cases} N_5 & \text{if } \pi = -1, \\ N_6 & \text{if } \pi = 1. \end{cases} & \kappa_{\mathbb{K}_7} &= \begin{cases} N_7 & \text{if } \pi = -1, \\ N_4 & \text{if } \pi = 1. \end{cases} \end{aligned}$$

- (4) Assume that $\left(\frac{p_1}{p_2}\right) = -1$ and $q = 2$, so

$$\begin{aligned} \kappa_{\mathbb{K}_4} &= \begin{cases} N_4 & \text{if } \pi = -1, \\ N_7 & \text{if } \pi = 1. \end{cases} & \kappa_{\mathbb{K}_5} &= \begin{cases} N_6 & \text{if } \pi = -1, \\ N_5 & \text{if } \pi = 1. \end{cases} \\ \kappa_{\mathbb{K}_6} &= \begin{cases} N_5 & \text{if } \pi = -1, \\ N_6 & \text{if } \pi = 1. \end{cases} & \kappa_{\mathbb{K}_7} &= \begin{cases} N_7 & \text{if } \pi = -1, \\ N_4 & \text{if } \pi = 1. \end{cases} \end{aligned}$$

To check the tables inputs, we use the following relations:

- $\sigma = \left(\frac{L/\mathbb{K}_3}{\mathfrak{P}} \right) = \left(\frac{L/\mathbb{k}}{N_{\mathbb{K}_3/\mathbb{k}}(\mathfrak{P})} \right) = \left(\frac{L/\mathbb{k}}{\mathcal{H}_1\mathcal{H}_2} \right)$, since $N_{\mathbb{K}_3/\mathbb{k}}(\mathfrak{P}) \sim \mathcal{H}_1\mathcal{H}_2$.
- $\tau = \left(\frac{L/\mathbb{K}_3}{\mathfrak{A}} \right) = \left(\frac{L/\mathbb{k}}{N_{\mathbb{K}_3/\mathbb{k}}(\mathfrak{A})} \right) = \left(\frac{L/\mathbb{k}}{\mathcal{H}_0} \right)$, because $N_{\mathbb{K}_3/\mathbb{k}}(\mathfrak{A}) \sim \mathcal{H}_0$.
- $\rho = \left(\frac{L/\mathbb{k}}{\mathcal{H}_1} \right)$.

Recall that the Artin map ϕ induces the following commutative diagram:

$$\begin{array}{ccc} \mathbf{Cl}_2(\mathbb{k}) & \xrightarrow{\phi} & G/G' \\ j_{\mathbb{K}_j/\mathbb{k}} \downarrow & & \downarrow V_{G/G_j} \\ \mathbf{Cl}_2(\mathbb{K}_j) & \xrightarrow{\phi} & G_j/G'_j \end{array}$$

the rows are isomorphisms and $V_{G/G_j} : G/G' \rightarrow G_j/G'_j$ is the group transfer map (Verlagerung) which has the following simple characterization when G_j is of index 2 in G . Let $G = G_j \cup zG_j$, then

$$V_{G/G_j}(gG') = \begin{cases} gz^{-1}gz.G'_j = g^2[g, z].G'_j & \text{if } g \in G_j; \\ g^2G'_j & \text{if } g \notin G_j. \end{cases}$$

Thus $\kappa_{\mathbb{K}_j} = \ker j_{\mathbb{K}_j/\mathbb{k}}$ is determined by $\ker V_{G/G_j}$.

(a) Consider the extension \mathbb{K}_1 ; we know that $G = \langle \sigma, \tau, \rho \rangle$ and, according to the table 1,

$$N_1 = \begin{cases} \langle [\mathcal{H}_0\mathcal{H}_1], [\mathcal{H}_0\mathcal{H}_2] \rangle = \langle [\mathcal{H}_1\mathcal{H}_2], [\mathcal{H}_0\mathcal{H}_1] \rangle & \text{if } \left(\frac{p_1}{p_2} \right) = 1, \\ \langle [\mathcal{H}_1], [\mathcal{H}_2] \rangle = \langle [\mathcal{H}_1\mathcal{H}_2], [\mathcal{H}_1] \rangle & \text{if } \left(\frac{p_1}{p_2} \right) = -1. \end{cases}$$

$$\text{Thus } G_1 = \text{Gal}(L/\mathbb{K}_1) = \begin{cases} \langle \sigma, \tau\rho, G' \rangle = \langle \sigma, \tau\rho, \tau^2 \rangle & \text{if } \left(\frac{p_1}{p_2} \right) = 1, \\ \langle \sigma, \rho, G' \rangle = \langle \sigma, \rho, \tau^2 \rangle = \langle \sigma, \rho \rangle & \text{if } \left(\frac{p_1}{p_2} \right) = -1. \end{cases}$$

This implies that $G/G_1 = \langle \tau \rangle = \{1, \tau G_1\}$; as

$$[\tau\rho, \sigma] = [\rho, \sigma] = \begin{cases} \sigma^2 & \text{if } q = 1, \\ \sigma^{-2m+2} = \sigma^{-2} & \text{if } q = 2, \end{cases} \quad \text{and } [\tau\rho, \tau^2] = \tau^4.$$

$$\text{So } G'_1 = \begin{cases} \langle \tau^4, \sigma^2 \rangle & \text{if } \left(\frac{p_1}{p_2} \right) = 1, \\ \langle \sigma^2 \rangle & \text{if } \left(\frac{p_1}{p_2} \right) = -1; \end{cases} \quad \text{from which we deduce that}$$

$$\text{Gal}(L/\mathbb{K}_1) = G_1/G'_1 \simeq \begin{cases} (2, 2, 2) & \text{if } \left(\frac{p_1}{p_2} \right) = 1, \\ (2, 4) & \text{if } \left(\frac{p_1}{p_2} \right) = -1, \end{cases} \quad \text{since } (\tau\rho)^2 = \rho^2 \in G'_1.$$

Compute the kernel of V_{G/G_1} .

- * $V_{G/G_1}(\sigma G') = \sigma^2[\sigma, \rho]G'_1 = \sigma^2\sigma^{-2}G'_1$ ou $\sigma^4G'_1 = G'_1$.
- * $V_{G/G_1}(\tau G') = \tau^2G'_1 \neq G'_1$.
- * $V_{G/G_1}(\rho G') = \rho^2G'_1 = G'_1$.

Consequently

$$\ker V_{G/G_1} = \langle \sigma G', \rho G' \rangle,$$

thus

$$\kappa_{\mathbb{K}_1} = \langle [\mathcal{H}_1\mathcal{H}_2], [\mathcal{H}_1] \rangle = \langle [\mathcal{H}_1], [\mathcal{H}_2] \rangle.$$

(b) For \mathbb{K}_2 , we proceed similarly, we get

$$G_2 = \text{Gal}(L/\mathbb{K}_2) = \begin{cases} \langle \sigma, \rho, \tau^2 \rangle & \text{if } \left(\frac{p_1}{p_2}\right) = 1, \\ \langle \sigma, \tau \rho \rangle & \text{if } \left(\frac{p_1}{p_2}\right) = -1; \end{cases} \quad \text{this implies that}$$

$$G'_2 = \begin{cases} \langle \tau^4, \sigma^2 \rangle & \text{if } \left(\frac{p_1}{p_2}\right) = 1, \\ \langle \sigma^2 \rangle & \text{if } \left(\frac{p_1}{p_2}\right) = -1; \end{cases} \quad \text{from which we deduce that}$$

$$\text{Gal}(L/\mathbb{K}_2) = G_2/G'_2 \simeq \begin{cases} (2, 2, 2) & \text{if } \left(\frac{p_1}{p_2}\right) = 1, \\ (2, 4) & \text{if } \left(\frac{p_1}{p_2}\right) = -1. \end{cases}$$

Thus

- * $V_{G/G_2}(\sigma G') = \sigma^2[\sigma, \rho]G'_2 = \sigma^2\sigma^{-2}G'_2$ or $\sigma^4G'_2 = G'_2$.
- * $V_{G/G_2}(\tau G') = \tau^2G'_2 \neq G'_2$.
- * $V_{G/G_2}(\rho G') = \rho^2[\rho, \tau]G'_2 = \tau^2G'_2 \neq G'_2$.
- * $V_{G/G_2}(\tau \rho G') = (\tau \rho)^2G'_2 = \rho^2G'_2 = G'_2$.

Therefore

$$\ker V_{G/G_2} = \langle \sigma G', \tau \rho G' \rangle,$$

hence

$$\kappa_{\mathbb{K}_2} = \langle [\mathcal{H}_1\mathcal{H}_2], [\mathcal{H}_0\mathcal{H}_1] \rangle = \langle [\mathcal{H}_0\mathcal{H}_1], [\mathcal{H}_0\mathcal{H}_2] \rangle.$$

(c) Take $\mathbb{K}_4 = \mathbb{k}(\sqrt{\pi_1\pi_3})$ and assume $\left(\frac{p_1}{p_2}\right) = 1$. We have to consider the following two cases:

1st case: Suppose that $\left(\frac{p_1}{p_2}\right)_4 \left(\frac{p_2}{p_1}\right)_4 = -1$, then $n = 1$, $m \geq 3$, $q = 1$, $N(\varepsilon_{p_1p_2}) = 1$ and, from Lemma 2, $\left(\frac{\pi_1}{\pi_3}\right) = -1$; this in turn has two sub-cases:

(α) If $\left(\frac{1+i}{\pi_1}\right) \left(\frac{1+i}{\pi_3}\right) = 1$, then Table 1 implies that

$$N_{\mathbb{K}_4/\mathbb{k}}(\mathbf{Cl}_2(\mathbb{K}_4)) = \langle [\mathcal{H}_0], [\mathcal{H}_2] \rangle. \text{ As } \left(\frac{L/\mathbb{k}}{\mathcal{H}_2}\right) = \left(\frac{L/\mathbb{k}}{\mathcal{H}_1}\right)^{-1} \left(\frac{L/\mathbb{k}}{\mathcal{H}_1\mathcal{H}_2}\right) = \rho^{-1}\sigma, \text{ so}$$

$$G_4 = \text{Gal}(L/\mathbb{K}_4) = \langle \tau, \rho^{-1}\sigma, G' \rangle.$$

On the other hand, in this sub-case we have $\rho^{-1}\sigma\rho = \sigma^{-1}$, thus $\rho^{-1}\sigma = (\rho\sigma)^{-1}$, therefore

$$G_4 = \text{Gal}(L/\mathbb{K}_4) = \langle \tau, \rho\sigma, \sigma^2 \rangle.$$

Since $[\rho\sigma, \sigma^2] = \sigma^4$, $[\rho\sigma, \tau] = \tau^2$ and $[\sigma, \tau^2] = 1$, so $G'_4 = \langle \tau^2, \sigma^4 \rangle$. From which we deduce that $G_4/G'_4 \simeq (2, 2, 2)$, since $(\rho\sigma)^2 = \rho^2 = \sigma^{2^{m-1}}$. Moreover $G/G_4 = \langle \sigma \rangle$, then:

- * $V_{G/G_4}(\sigma G') = \sigma^2 G'_4 \neq G'_4$.
- * $V_{G/G_4}(\tau G') = \tau^2 [\tau, \sigma] G'_4 = G'_4$.
- * $V_{G/G_4}(\rho G') = \rho^2 G'_4 = G'_4$.

Consequently

$$\ker V_{G/G_4} = \langle \tau G', \rho G' \rangle,$$

and thus

$$\kappa_{\mathbb{K}_4} = \langle [\mathcal{H}_0], [\mathcal{H}_1] \rangle.$$

(β) If $\left(\frac{1+i}{\pi_1}\right) \left(\frac{1+i}{\pi_3}\right) = -1$, similarly we get

$$N_4 = \langle [\mathcal{H}_2], [\mathcal{H}_0\mathcal{H}_1] \rangle = \langle [\mathcal{H}_0\mathcal{H}_1\mathcal{H}_2], [\mathcal{H}_0\mathcal{H}_1] \rangle,$$

thus

$$G_4 = \text{Gal}(L/\mathbb{K}_4) = \langle \sigma\tau, \tau\rho, \sigma^2 \rangle.$$

As $[\tau\rho, \sigma\tau] = [\rho, \tau][\rho, \sigma] = (\sigma\tau)^2$ and $[\tau\rho, \sigma^2] = \sigma^4$, so $G'_4 = \langle (\sigma\tau)^2, \sigma^4 \rangle$, since $\sigma^4 = \sigma^4\tau^4 = (\sigma\tau)^4$. It is clear that $(\sigma\tau)^2$, σ^4 and $(\tau\rho)^2 = \rho^2 = (\tau\sigma)^{2^{m-1}}$ are in G'_4 ; hence $G_4/G'_4 \simeq (2, 2, 2)$.

Let us compute the kernel of V_{G/G_4} . Since $G/G_4 = \langle \sigma \rangle$, then:

- * $V_{G/G_4}(\sigma G') = \sigma^2 G'_4 \neq G'_4$.
- * $V_{G/G_4}(\sigma\tau G') = (\sigma\tau)^2 [\sigma\tau, \sigma] G'_4 = G'_4$.
- * $V_{G/G_4}(\rho G') = \rho^2 G'_4 = G'_4$.

This implies that

$$\ker V_{G/G_4} = \langle \rho G', \sigma\tau G' \rangle,$$

hence

$$\kappa_{\mathbb{K}_2} = \langle [\mathcal{H}_1], [\mathcal{H}_1\mathcal{H}_2\mathcal{H}_0] \rangle = \langle [\mathcal{H}_0\mathcal{H}_2], [\mathcal{H}_1] \rangle.$$

- 2nd case: Suppose $\left(\frac{p_1}{p_2}\right)_4 \left(\frac{p_2}{p_1}\right)_4 = 1$, then $m = 2$, $n \geq 2$ and, from Lemma 2, $\left(\frac{\pi_1}{\pi_3}\right) = 1$. Similarly, there are two sub-cases to distinguish:

(α) If $\left(\frac{1+i}{\pi_1}\right)\left(\frac{1+i}{\pi_3}\right) = 1$, then the table 1 implies that $N_4 = \langle [\mathcal{H}_0], [\mathcal{H}_1] \rangle$, from which we deduce that

$$G_4 = \text{Gal}(L/\mathbb{K}_4/) = \langle \tau, \rho, G' \rangle = \langle \tau, \rho, \sigma^2 \rangle;$$

as $\rho^2 = \sigma^2$ or $\rho^2 = \sigma^2 \tau^{2^n}$, so

$$G_4 = \text{Gal}(L/\mathbb{K}_4/) = \langle \tau, \rho \rangle.$$

We know that $[\rho, \tau] = \tau^2$, thus $G'_4 = \langle \tau^2 \rangle$, this in turn yields that $G_4/G'_4 \simeq (2, 4)$, since $\rho^4 = 1$. Moreover $G/G_4 = \langle \sigma \rangle$, so:

$$\begin{aligned} * \quad & V_{G/G_4}(\sigma G') = \sigma^2 G'_4 \neq G'_4. \\ * \quad & V_{G/G_4}(\tau G') = \tau^2 [\tau, \sigma] G'_4 = G'_4. \\ * \quad & V_{G/G_4}(\rho G') = \rho^2 [\rho, \sigma] G'_4 = \begin{cases} \rho^2 \sigma^2 G'_4 = \sigma^4 G'_4 = G'_4, & \text{if } q = 1, \\ \rho^2 \sigma^{-2} G'_4 = \sigma^2 \sigma^{-2} \tau^{2^n} G'_4 = G'_4, & \text{if } q = 2; \end{cases} \end{aligned}$$

Consequently

$$\ker V_{G/G_4} = \langle \tau G', \rho G' \rangle,$$

and thus

$$\kappa_{\mathbb{K}_4} = \langle [\mathcal{H}_0], [\mathcal{H}_1] \rangle.$$

(β) If $\left(\frac{1+i}{\pi_1}\right)\left(\frac{1+i}{\pi_3}\right) = -1$, then we get

$$N_4 = \langle [\mathcal{H}_1], [\mathcal{H}_0 \mathcal{H}_2] \rangle = \langle [\mathcal{H}_1 \mathcal{H}_2 \mathcal{H}_0], [\mathcal{H}_1] \rangle,$$

hence

$$G_4 = \text{Gal}(L/\mathbb{K}_4/) = \langle \sigma \tau, \rho, \sigma^2, \tau^2 \rangle = \langle \sigma \tau, \rho \rangle.$$

$$\text{As } [\rho, \sigma \tau] = [\rho, \tau][\rho, \sigma] = \begin{cases} (\sigma \tau)^2 & \text{if } q = 1, \\ \sigma^{-2} \tau^2 & \text{if } q = 2, \end{cases}$$

$$\text{so } G'_4 = \begin{cases} \langle (\sigma \tau)^2 \rangle & \text{if } q = 1, \\ \langle \sigma^{-2} \tau^2 \rangle & \text{if } q = 2. \end{cases}$$

Moreover, as $\sigma^{-2} \tau^2 = \sigma^{-4} \sigma^2 \tau^2 = \tau^{-2^{n+1}} (\sigma \tau)^2 = (\sigma \tau)^{2+2^{n+1}}$, so $G'_4 = \langle (\sigma \tau)^2 \rangle$, thus $G_4/G'_4 \simeq (2, 4)$. On the other hand, $G/G_4 = \langle \tau \rangle$, hence:

$$\begin{aligned} * \quad & V_{G/G_4}(\sigma G') = \sigma^2 G'_4 \neq G'_4. \\ * \quad & V_{G/G_4}(\tau G') = \tau^2 G'_4 \neq G'_4. \\ * \quad & V_{G/G_4}(\rho G') = \rho^2 [\rho, \tau] G'_4 = \rho^2 \tau^2 G'_4 = \\ & \begin{cases} (\sigma \tau)^2 G'_4 = G'_4 & \text{if } q = 1, \\ \sigma^2 \tau^{2^n} \tau^2 G'_4 = \sigma^2 \tau^{-4} \tau^2 G'_4 = \sigma^2 \tau^{-2} G'_4 = G'_4 & \text{if } q = 2; \end{cases} \\ & \text{since in the second case } (q = 2) \text{ we have } \tau^{2^{n+2}} = 1, \text{ thus } \tau^{2^n} = \tau^{-4}. \end{aligned}$$

$$* \quad V_{G/G_4}(\sigma\tau G') = (\sigma\tau)^2[\sigma\tau, \tau]G'_4 = G'_4.$$

Therefore

$$\ker V_{G/G_4} = \langle \rho G', \sigma\tau G' \rangle,$$

from which we deduce that

$$\kappa_{\mathbb{K}_4} = \langle [\mathcal{H}_1], [\mathcal{H}_1\mathcal{H}_2\mathcal{H}_0] \rangle = \langle [\mathcal{H}_0\mathcal{H}_2], [\mathcal{H}_1] \rangle.$$

Conclusion: Let $\mathbb{K}_4 = \mathbb{k}(\sqrt{\pi_1\pi_3})$ and $G_4 = \text{Gal}(L/\mathbb{K}_4)$. Then $\mathbf{Cl}_2(\mathbb{K}_4)$ is of type $(2, 2, 2)$ if $\left(\frac{p_1}{p_2}\right)_4 \left(\frac{p_2}{p_1}\right)_4 = -1$, and of type $(2, 4)$ otherwise. Moreover

- (i) If $\left(\frac{1+i}{\pi_1}\right) \left(\frac{1+i}{\pi_3}\right) = 1$, then $\ker V_{G/G_4} = \langle \tau G', \rho G' \rangle$ and $\kappa_{\mathbb{K}_4} = \langle [\mathcal{H}_0], [\mathcal{H}_1] \rangle$.
- (ii) If $\left(\frac{1+i}{\pi_1}\right) \left(\frac{1+i}{\pi_3}\right) = -1$, then $\ker V_{G/G_4} = \langle \rho G', \sigma\tau G' \rangle$, and $\kappa_{\mathbb{K}_4} = \langle [\mathcal{H}_0\mathcal{H}_2], [\mathcal{H}_1] \rangle$.

Assume now that $\left(\frac{p_1}{p_2}\right) = -1$. We have also two cases to distinguish:

1st case: Suppose $q = 1$, so Lemma 6 yields that $n = 1$, $m \geq 3$. Then we need to consider two sub-cases:

- If $\left(\frac{\pi_1}{\pi_3}\right) = 1$, then $\left(\frac{1+i}{\pi_1}\right) \left(\frac{1+i}{\pi_3}\right) = 1$, hence

$$N_4 = \langle [\mathcal{H}_0], [\mathcal{H}_2] \rangle \text{ and } G_4 = \text{Gal}(L/\mathbb{K}_4) = \langle \tau, \rho\sigma, \sigma^2 \rangle.$$

So $G'_4 = \langle \tau^2, \sigma^4 \rangle$, which involves that $G_4/G'_4 \simeq (2, 2, 2)$, since $(\rho\sigma)^2 = \rho^2 = \tau^2\sigma^{2^{m-1}}$. Moreover $G/G_4 = \langle \sigma \rangle$, then:

- * $V_{G/G_4}(\sigma G') = \sigma^2 G'_4 \neq G'_4$.
- * $V_{G/G_4}(\tau G') = \tau^2[\tau, \sigma]G'_4 = G'_4$.
- * $V_{G/G_4}(\rho G') = \rho^2 G'_4 = G'_4$.

Consequently

$$\ker V_{G/G_4} = \langle \tau G', \rho G' \rangle,$$

and thus

$$\kappa_{\mathbb{K}_2} = \langle [\mathcal{H}_0], [\mathcal{H}_1] \rangle.$$

- If $\left(\frac{\pi_1}{\pi_3}\right) = -1$, then $\left(\frac{1+i}{\pi_1}\right) \left(\frac{1+i}{\pi_3}\right) = -1$, similarly we get

$$N_4 = \langle [\mathcal{H}_1], [\mathcal{H}_0\mathcal{H}_2] \rangle = \langle [\mathcal{H}_0\mathcal{H}_1\mathcal{H}_2], [\mathcal{H}_1] \rangle,$$

thus

$$G_4 = \text{Gal}(L/\mathbb{K}_4) = \langle \sigma\tau, \rho \rangle.$$

So $G'_4 = \langle (\sigma\tau)^2 \rangle$, hence $G_4/G'_4 \simeq (2, 4)$. As $G/G_4 = \langle \tau \rangle$, thus

- * $V_{G/G_4}(\sigma G') = \sigma^2 G'_4 \neq G'_4$.

- * $V_{G/G_4}(\sigma\tau G') = (\sigma\tau)^2[\sigma\tau, \sigma]G'_4 = G'_4.$
- * $V_{G/G_4}(\rho G') = \rho^2[\rho, \tau]G'_4 = \rho^2\tau^2G'_4 = G'_4.$

This implies that

$$\ker V_{G/G_4} = \langle \rho G', \sigma\tau G' \rangle,$$

hence

$$\kappa_{\mathbb{K}_2} = \langle [\mathcal{H}_1], [\mathcal{H}_1\mathcal{H}_2\mathcal{H}_0] \rangle = \langle [\mathcal{H}_0\mathcal{H}_2], [\mathcal{H}_1] \rangle.$$

Conclusion. Let $\mathbb{K}_4 = \mathbb{k}(\sqrt{\pi_1\pi_3})$; assume that $\left(\frac{p_1}{p_2}\right) = -1$ and $q = 1$. If $\left(\frac{\pi_1}{\pi_3}\right) = -1$, then $\mathbf{Cl}_2(\mathbb{K}_4)$ is of type $(2, 4)$ and of type $(2, 2, 2)$ otherwise. Moreover

- (i) If $\left(\frac{\pi_1}{\pi_3}\right) = 1$, then $\ker V_{G/G_4} = \langle \rho G', \tau G' \rangle$ and $\kappa_{\mathbb{K}_4} = \langle [\mathcal{H}_0], [\mathcal{H}_1] \rangle.$
- (ii) If $\left(\frac{\pi_1}{\pi_3}\right) = -1$, then $\ker V_{G/G_4} = \langle \rho G', \sigma\tau G' \rangle$ and $\kappa_{\mathbb{K}_4} = \langle [\mathcal{H}_1], [\mathcal{H}_0\mathcal{H}_2] \rangle.$

2nd case: Suppose $q = 2$, so, according to Lemma 6, $n = 1$ and $m = 2$. Then we have two sub-cases to consider :

- If $\left(\frac{\pi_1}{\pi_3}\right) = 1$, then $\left(\frac{1+i}{\pi_1}\right)\left(\frac{1+i}{\pi_3}\right) = -1$, hence

$$N_4 = \langle [\mathcal{H}_2], [\mathcal{H}_0\mathcal{H}_1] \rangle \text{ and } G_4 = \text{Gal}(L/\mathbb{K}_4/) = \langle \sigma\tau, \tau\rho, \sigma^2 \rangle.$$

So $G'_4 = \langle \rho^2, \sigma^4 \rangle$, since $\rho^2 = \tau^2\sigma^2$; which implies that $G_4/G'_4 \simeq (2, 2, 2)$. Moreover $G/G_4 = \langle \rho \rangle$, then:

- * $V_{G/G_4}(\sigma G') = \sigma^2 G'_4 \neq G'_4.$
- * $V_{G/G_4}(\tau G') = \tau^2 G'_4 \neq G'_4.$
- * $V_{G/G_4}(\rho G') = \rho^2 G'_4 = G'_4.$
- * $V_{G/G_4}(\sigma\tau G') = (\sigma\tau)^2[\sigma\tau, \rho]G'_4 = (\sigma\tau)^2(\sigma\tau)^{-2}\sigma^4 G'_4 = G'_4.$

Consequently

$$\ker V_{G/G_4} = \langle \rho G', \sigma\tau G' \rangle,$$

thus

$$\kappa_{\mathbb{K}_2} = \langle [\mathcal{H}_1], [\mathcal{H}_0\mathcal{H}_2] \rangle.$$

- If $\left(\frac{\pi_1}{\pi_3}\right) = -1$, then $\left(\frac{1+i}{\pi_1}\right)\left(\frac{1+i}{\pi_3}\right) = 1$, similarly we get

$$N_4 = \langle [\mathcal{H}_1], [\mathcal{H}_0] \rangle \text{ and } G_4 = \text{Gal}(L/\mathbb{K}_4/) = \langle \tau, \rho \rangle.$$

So $G'_4 = \langle \tau^2 \rangle$, hence $G_4/G'_4 \simeq (2, 4)$. As $G/G_4 = \langle \sigma \rangle$, thus

- * $V_{G/G_4}(\sigma G') = \sigma^2 G'_4 \neq G'_4.$
- * $V_{G/G_4}(\tau G') = \tau^2 G'_4 = G'_4.$
- * $V_{G/G_4}(\rho G') = \rho^2[\rho, \sigma]G'_4 = \rho^2\sigma^{-2}G'_4 = G'_4.$

This implies that

$$\ker V_{G/G_4} = \langle \rho G', \tau G' \rangle,$$

hence

$$\kappa_{\mathbb{K}_2} = \langle [\mathcal{H}_1], [\mathcal{H}_0] \rangle.$$

Conclusion. Let $\mathbb{K}_4 = \mathbb{k}(\sqrt{\pi_1 \pi_3})$; assume that $\left(\frac{p_1}{p_2}\right) = -1$ and $q = 2$. Then $\mathbf{Cl}_2(\mathbb{K}_4)$ is of type $(2, 4)$ if $\left(\frac{\pi_1}{\pi_3}\right) = -1$, and of type $(2, 2, 2)$ otherwise. Moreover

- (i) If $\left(\frac{\pi_1}{\pi_3}\right) = -1$, then $\ker V_{G/G_4} = \langle \rho G', \tau G' \rangle$ and $\kappa_{\mathbb{K}_4} = \langle [\mathcal{H}_0], [\mathcal{H}_1] \rangle$.
- (ii) Else $\ker V_{G/G_4} = \langle \rho G', \sigma \tau G' \rangle$ and $\kappa_{\mathbb{K}_4} = \langle [\mathcal{H}_1], [\mathcal{H}_0 \mathcal{H}_2] \rangle$.

Proceeding similarly we show the other tables inputs.

4.2.3. Capitulation kernels $\kappa_{\mathbb{L}_j}$ and $\text{Gal}(\mathbb{k}_2^{(2)}/\mathbb{L}_j)$. From the subsection 4.2.2, we deduce that $\kappa_{\mathbb{L}_j} = \mathbf{Cl}_2(\mathbb{k})$. In what follows, we compute the Galois groups $\mathcal{G}_j = \text{Gal}(\mathbb{k}_2^{(2)}/\mathbb{L}_j)$, their derived groups \mathcal{G}'_j and the abelian type invariants of $\mathbf{Cl}_2(\mathbb{L}_j)$. The results are summarized in the following tables; note that the left hand side of columns (if it exists) refers to the case $\left(\frac{1+i}{\pi_1}\right)\left(\frac{1+i}{\pi_3}\right) = 1$, the right hand side to the case $\left(\frac{1+i}{\pi_1}\right)\left(\frac{1+i}{\pi_3}\right) = -1$.

$$\text{Put } \begin{cases} \alpha = \min(n, m) \text{ and } \beta = \max(n+1, m+1), \\ a = \min(n, m-1) \text{ and } b = \max(n+1, m), \\ \pi = \left(\frac{\pi_1}{\pi_3}\right), \\ B = \left(\frac{1+i}{\pi_1}\right)\left(\frac{1+i}{\pi_3}\right). \end{cases}$$

Table 4: Invariants of $\mathbf{Cl}_2(\mathbb{L}_j)$ for the case $\left(\frac{p_1}{p_2}\right) = 1$

\mathbb{L}_j	\mathcal{G}_j	\mathcal{G}'_j	$Cl_2(\mathbb{L}_j)$
\mathbb{L}_1	$\langle \tau^2, \sigma \rangle$	$\langle 1 \rangle$	$(2^n, 2^m)$ if $q = 1$ $(2^\alpha, 2^\beta)$ if $q = 2$
\mathbb{L}_2	$\pi = -1$ $\pi = 1$	$\langle \sigma \tau \rho, \sigma^2, \tau^2 \rangle$ $\langle \tau \rho, \sigma^2, \tau^2 \rangle$ $\langle \sigma \tau \rho, \tau^2 \rangle$ $\langle \tau^4 \rangle$	$\langle \sigma^4, \tau^4 \rangle$ $(2, 2, 2)$ $(2, 4)$
\mathbb{L}_3	$\pi = -1$ $\pi = 1$	$\langle \tau \rho, \sigma^2, \tau^2 \rangle$ $\langle \sigma \tau \rho, \sigma^2, \tau^2 \rangle$ $\langle \sigma \tau \rho, \tau^2 \rangle$ $\langle \tau^4 \rangle$	$\langle \sigma^4, \tau^4 \rangle$ $(2, 2, 2)$ $(2, 4)$
\mathbb{L}_4	$\pi = -1$ $\pi = 1$	$\langle \sigma \rho, \sigma^2, \tau^2 \rangle$ $\langle \rho, \tau^2 \rangle$	$\langle \sigma^4, \tau^4 \rangle$ $(2, 2, 2)$ $(2, 4)$
\mathbb{L}_5	$\pi = -1$ $\pi = 1$	$\langle \rho, \sigma^2, \tau^2 \rangle$ $\langle \rho \sigma, \tau^2 \rangle$	$\langle \sigma^4, \tau^4 \rangle$ $(2, 2, 2)$ $(2, 4)$
\mathbb{L}_6	$B = -1$ $B = 1$	$\langle \sigma \tau, \sigma^2 \rangle$ $\langle \tau, \sigma^2 \rangle$	$\langle 1 \rangle$ $\langle 1 \rangle$ $\left\{ \begin{array}{l} (2^a, 2^b) \text{ if } q = 1 \\ (2, 2^{n+2}) \text{ if } q = 2 \\ (2^{m-1}, 2^{n+1}) \text{ if } q = 1 \\ (2, 2^{n+2}) \text{ if } q = 2 \end{array} \right.$

\mathbb{L}_j	\mathcal{G}_j	\mathcal{G}_j	$\mathbf{Cl}_2(\mathbb{L}_j)$
$B = -1$	$\langle \tau, \sigma^2 \rangle$	$\langle 1 \rangle$	$\begin{cases} (2^{m-1}, 2^{n+1}) & \text{if } q = 1 \\ (2, 2^{n+2}) & \text{if } q = 2 \end{cases}$
$B = 1$	$\langle \sigma\tau, \sigma^2 \rangle$	$\langle 1 \rangle$	$\begin{cases} (2^a, 2^b) & \text{if } q = 1 \\ (2, 2^{n+2}) & \text{if } q = 2 \end{cases}$

For the following table, the left hand side of columns (if it exists) refers to the case $(\frac{\pi_1}{\pi_3}) = -1$, the right hand side refers to the case $(\frac{\pi_1}{\pi_2}) = 1$. Recall that if

$$\left(\frac{p_1}{p_2}\right) = -1, \text{ then } n = 1 \text{ and } \begin{cases} m = 2 & \text{if } q = 2, \\ m \geq 2 & \text{if } q = 1 \end{cases}$$

Table 5: Invariants of $\mathbf{Cl}_2(\mathbb{L}_j)$ for the case $\left(\frac{p_1}{p_2}\right) = -1$

\mathbb{L}_j	\mathcal{G}_j	\mathcal{G}'_j	$\mathbf{Cl}_2(\mathbb{L}_j)$
\mathbb{L}_1	$\langle \tau^2, \sigma \rangle$	$\langle 1 \rangle$	$(2^n, 2^m)$ if $q = 1$ $(2^n, 2^{m+1})$ if $q = 2$
\mathbb{L}_2	$\langle \rho, \sigma^2 \rangle \langle \rho \sigma, \sigma^2 \rangle$	$\langle \sigma^4 \rangle$	$(2, 4)$
\mathbb{L}_3	$\langle \rho \sigma, \sigma^2 \rangle \langle \rho, \sigma^2 \rangle$	$\langle \sigma^4 \rangle$	$(2, 4)$
\mathbb{L}_4 $q = 1$ $q = 2$	$\langle \sigma \tau \rho, \sigma^2 \rangle$ $\langle \tau \rho, \sigma^2 \rangle$	$\langle \sigma^4 \rangle$	$(2, 4)$
\mathbb{L}_5 $q = 1$ $q = 2$	$\langle \tau \rho, \sigma^2 \rangle$ $\langle \sigma \tau \rho, \sigma^2 \rangle$	$\langle \sigma^4 \rangle$	$(2, 4)$
\mathbb{L}_6 $q = 1$ $q = 2$	$\langle \sigma \tau, \tau^2 \rangle \langle \tau, \sigma^2 \rangle$ $\langle \tau, \sigma^2 \rangle \langle \sigma \tau, \sigma^2 \rangle$	$\langle 1 \rangle$ $\langle 1 \rangle$	$(2, 2^m) (4, 2^{m-1})$ $(2, 8) (4, 4)$
\mathbb{L}_7 $q = 1$ $q = 2$	$\langle \tau, \sigma^2 \rangle \langle \sigma \tau, \sigma^2 \rangle$ $\langle \sigma \tau, \sigma^2 \rangle \langle \tau, \sigma^2 \rangle$	$\langle 1 \rangle$ $\langle 1 \rangle$	$(4, 2^{m-1}) (2, 2^m)$ $(4, 4) (2, 8)$

Check the entries in some cases.

* Take $\mathbb{L}_1 = \mathbb{k}^{(*)} = \mathbb{K}_1.\mathbb{K}_2.\mathbb{K}_3$. Since $\text{Gal}(L/\mathbb{L}_1) = \mathcal{G}_1 = G_1 \cap G_2$, then

$\mathcal{G}_1 = \langle \sigma, \tau\rho, \tau^2 \rangle \cap \langle \sigma, \rho, \tau^2 \rangle = \langle \sigma, \tau^2 \rangle$, thus $\mathcal{G}'_1 = \langle 1 \rangle$. As

$$\begin{cases} \sigma^{2^m} = \tau^{2^{n+1}} = 1 & \text{if } q = 1, \\ \sigma^{2^m} = \tau^{2^{n+1}} \text{ and } \sigma^{2^{m+1}} = \tau^{2^{n+2}} = 1 & \text{if } q = 2, \end{cases}$$

$$\text{so } \mathbf{Cl}_2(\mathbb{L}_1) \simeq \begin{cases} (2^n, 2^m) & \text{if } q = 1, \\ (2^{\min(n,m)}, 2^{\max(n+1,m+1)}) & \text{if } q = 2. \end{cases}$$

* Take $\mathbb{L}_2 = \mathbb{K}_1.\mathbb{K}_4.\mathbb{K}_6$ and assume that $(\frac{p_1}{p_2}) = 1$, then $\mathcal{G}_2 = \text{Gal}(L/\mathbb{L}_2) =$

$G_1 \cap G_4 \cap G_6$. There are two cases to distinguish:

- 1st case: If $\left(\frac{p_1}{p_2}\right)_4 \left(\frac{p_2}{p_1}\right)_4 = \left(\frac{\pi_1}{\pi_3}\right) = -1$, then $q = 1$ and

$$\mathcal{G}_2 = \begin{cases} \langle \sigma, \tau\rho, \tau^2 \rangle \cap \langle \sigma\tau, \rho, \tau^2 \rangle = \langle \sigma\tau\rho, \sigma^2, \tau^2 \rangle & \text{if } (\frac{1+i}{\pi_1})(\frac{1+i}{\pi_3}) = 1, \\ \langle \sigma, \tau\rho, \tau^2 \rangle \cap \langle \sigma\tau, \tau\rho, \sigma^2 \rangle = \langle \tau\rho, \sigma^2, \tau^2 \rangle & \text{if } (\frac{1+i}{\pi_1})(\frac{1+i}{\pi_3}) = -1, \end{cases}$$

thus $\mathcal{G}'_2 = \langle \sigma^4, \tau^4 \rangle$. On the other hand, as in this case $(\sigma\tau\rho)^2 = (\tau\rho)^2 = \rho^2 = \sigma^{2^{m-1}}$,

so $\mathbf{Cl}_2(\mathbb{L}_2) \simeq (2, 2, 2)$.

- 2nd case: If $\left(\frac{p_1}{p_2}\right)_4 \left(\frac{p_2}{p_1}\right)_4 = \left(\frac{\pi_1}{\pi_3}\right) = 1$, then

$$\mathcal{G}_2 = \begin{cases} \langle \sigma, \tau\rho, \tau^2 \rangle \cap \langle \tau, \rho, \sigma^2 \rangle = \langle \tau\rho, \sigma^2, \tau^2 \rangle & \text{if } \left(\frac{1+i}{\pi_1}\right)\left(\frac{1+i}{\pi_3}\right) = 1, \\ \langle \sigma, \tau\rho, \tau^2 \rangle \cap \langle \sigma\tau, \rho, \sigma^2 \rangle = \langle \sigma\tau\rho, \sigma^2, \tau^2 \rangle & \text{if } \left(\frac{1+i}{\pi_1}\right)\left(\frac{1+i}{\pi_3}\right) = -1, \end{cases}$$

as, in this case, $(\tau\rho)^2 = \rho^2$ and $(\sigma\tau\rho)^2 = \begin{cases} \rho^2 = \sigma^2 & \text{if } q = 1, \\ \rho^2\sigma^4 = \sigma^2\tau^{-2n} & \text{if } q = 2; \end{cases}$

since if $q = 2$, we have $\sigma^4 = \tau^{2n+1}$ and $\rho^2 = \sigma^2\tau^{2n}$,

$$\text{so } \mathcal{G}_2 = \begin{cases} \langle \tau\rho, \tau^2 \rangle & \text{if } \left(\frac{1+i}{\pi_1}\right)\left(\frac{1+i}{\pi_3}\right) = 1, \\ \langle \sigma\tau\rho, \tau^2 \rangle & \text{if } \left(\frac{1+i}{\pi_1}\right)\left(\frac{1+i}{\pi_3}\right) = -1, \end{cases}$$

we infer that $\mathcal{G}'_2 = \langle \tau^4 \rangle$. From which we deduce that $\mathbf{Cl}_2(\mathbb{L}_2) \simeq (2, 4)$, since $(\sigma\tau\rho)^4 = (\tau\rho)^4 = 1$.

Assume now that $\left(\frac{p_1}{p_2}\right) = -1$, then $\mathcal{G}_2 = \begin{cases} \langle \rho, \sigma^2 \rangle & \text{if } \left(\frac{\pi_1}{\pi_3}\right) = -1, \\ \langle \rho\sigma, \sigma^2 \rangle & \text{if } \left(\frac{\pi_1}{\pi_3}\right) = 1. \end{cases}$

Thus $\mathcal{G}'_2 = \langle \sigma^4 \rangle$, hence $\mathbf{Cl}_2(\mathbb{L}_2) \simeq (2, 4)$.

* Finally, we take $\mathbb{L}_6 = \mathbb{K}_3.\mathbb{K}_4.\mathbb{K}_7$ and we assume that $\left(\frac{p_1}{p_2}\right) = 1$, then $\text{Gal}(L/\mathbb{L}_6) = \mathcal{G}_6 = G_3 \cap G_4 \cap G_7$, which yields that

$$\mathcal{G}_6 = \begin{cases} \langle \tau, \sigma \rangle \cap \langle \tau, \rho, \sigma^2 \rangle = \langle \tau, \sigma^2 \rangle & \text{if } \left(\frac{1+i}{\pi_1}\right)\left(\frac{1+i}{\pi_3}\right) = 1, \\ \langle \tau, \sigma \rangle \cap \langle \sigma\tau, \rho, \sigma^2 \rangle = \langle \sigma\tau, \sigma^2 \rangle = \langle \sigma\tau, \tau^2 \rangle & \text{if } \left(\frac{1+i}{\pi_1}\right)\left(\frac{1+i}{\pi_3}\right) = -1, \end{cases}$$

therefore $\mathcal{G}'_6 = \langle 1 \rangle$.

As $\begin{cases} \sigma^{2^m} = \tau^{2^{n+1}} = 1 & \text{if } q = 1, \\ \sigma^{2^m} = \tau^{2^{n+1}} \text{ and } \sigma^{2^{m+1}} = \tau^{2^{n+2}} = 1 & \text{if } q = 2, \end{cases}$ so

- 1st case: If $q = 1$, then $\mathbf{Cl}_2(\mathbb{L}_6)$ is of type

$$\begin{cases} (2^{\min(n+1, m-1)}, 2^{\max(n+1, m-1)}) = (2^{n+1}, 2^{m-1}) & \text{if } \left(\frac{1+i}{\pi_1}\right)\left(\frac{1+i}{\pi_3}\right) = 1, \\ (2^{\min(n, m-1)}, 2^{\max(n+1, m)}) & \text{if } \left(\frac{1+i}{\pi_1}\right)\left(\frac{1+i}{\pi_3}\right) = -1, \end{cases}$$

- 2nd case: If $q = 2$, then $m = 2$, $n \geq 2$ and $\mathbf{Cl}_2(\mathbb{L}_6)$ is of type

$$\begin{cases} (2^{\min(n+1, m-1)}, 2^{\max(n+2, m+1)}) = (2, 2^{n+2}) & \text{if } \left(\frac{1+i}{\pi_1}\right)\left(\frac{1+i}{\pi_3}\right) = 1, \\ (2^{\min(n, m-1)}, 2^{\max(n+2, m+1)}) = (2, 2^{n+2}) & \text{if } \left(\frac{1+i}{\pi_1}\right)\left(\frac{1+i}{\pi_3}\right) = -1, \end{cases}$$

Assume that $\left(\frac{p_1}{p_2}\right) = -1$, so

- 1st case: If $q = 1$, then $\mathcal{G}_6 = \begin{cases} \langle \tau\sigma, \tau^2 \rangle & \text{if } \left(\frac{\pi_1}{\pi_3}\right) = -1, \\ \langle \tau, \sigma^2 \rangle & \text{if } \left(\frac{\pi_1}{\pi_3}\right) = 1, \end{cases}$

therefore $\mathcal{G}'_6 = \langle 1 \rangle$. So $\mathbf{Cl}_2(\mathbb{L}_6) \simeq \begin{cases} (2, 2^m) & \text{if } \left(\frac{\pi_1}{\pi_3}\right) = -1, \\ (4, 2^{m-1}) & \text{if } \left(\frac{\pi_1}{\pi_3}\right) = 1, \end{cases}$

- 2nd case: If $q = 2$, then $m = 2$, $n = 1$ and $\mathcal{G}_6 = \begin{cases} \langle \tau, \sigma^2 \rangle & \text{if } \left(\frac{\pi_1}{\pi_3}\right) = -1, \\ \langle \tau\sigma, \tau^2 \rangle & \text{if } \left(\frac{\pi_1}{\pi_3}\right) = 1. \end{cases}$

Hence $\mathbf{Cl}_2(\mathbb{L}_6) \simeq \begin{cases} (2, 8) = (2, 2^{n+2}) & \text{if } (\frac{\pi_1}{\pi_3}) = -1, \\ (4, 4) & \text{if } (\frac{\pi_1}{\pi_3}) = 1, \end{cases}$

The other tables entries are checked similarly.

5. NUMERICAL EXAMPLES

Table 6 gives the structure of the class group $\mathbf{Cl}(\mathbb{k})$ of the bicyclic biquadratic field $\mathbb{k} = \mathbb{Q}(\sqrt{2p_1p_2}, i)$, its discriminant $\text{disc}(\mathbb{k})$, the structures of the class groups of its two quadratic subfields k_0, \bar{k}_0 and the coclass of $G = \text{Gal}(\mathbb{k}_2^{(2)}/\mathbb{k})$. Tables 7 and 8 give the structures of the class groups $\mathbf{Cl}(\mathbb{K}_j)$. Tables 9 and 10 give the structures of the class groups $\mathbf{Cl}(\mathbb{L}_j)$. Finally, Tables 11 and 12 give the structures of the class groups of $\mathbf{Cl}(\mathbb{K}_j)$ and $\mathbf{Cl}(\mathbb{L}_j)$ for the case $(\frac{p_1}{p_2}) = -1$. Note that $\pi = (\frac{\pi_1}{\pi_3})$ and $b = \left(\frac{1+i}{\pi_1}\right) \left(\frac{1+i}{\pi_3}\right)$. Computation are made using PARI/GP [18].

Table 6: Invariants of \mathbb{k}

$d = p_1 \cdot p_2 \cdot q$	q	$(\frac{p_1}{p_2})$	m, n	$\mathbf{Cl}_2(k_0)$	$\mathbf{Cl}_2(\bar{k}_0)$	$\mathbf{Cl}_2(\mathbb{k})$	$\text{disc}(\mathbb{k})$	$\text{cc}(G)$
130 = 2.5.13	2	-1	2, 1	(2, 2)	(2, 2)	(2, 2, 2)	1081600	3
290 = 2.5.29	1	1	2, 2	(2, 2)	(10, 2)	(10, 2, 2)	5382400	3
370 = 2.5.37	1	-1	3, 1	(2, 2)	(6, 2)	(6, 2, 2)	8761600	3
754 = 2.13.29	1	1	3, 1	(2, 2)	(10, 2)	(10, 2, 2)	36385024	3
3922 = 2.53.37	1	1	4, 1	(2, 2)	(10, 2)	(10, 2, 2)	984453376	3
4610 = 2.461.5	2	1	2, 4	(2, 2)	(26, 2)	(26, 2, 2)	1360134400	3
5122 = 2.197.13	1	-1	5, 1	(2, 2)	(14, 2)	(14, 2, 2)	1679032576	3
5410 = 2.5.541	1	1	2, 3	(2, 2)	(22, 2)	(22, 2, 2)	1873158400	3

Table 7: Invariants of \mathbb{K}_j for the case $(\frac{p_1}{p_2}) = 1$ and $(\frac{\pi_1}{\pi_3}) = -1$

$2 \cdot p_1 \cdot p_2$	q	m, n	$\mathbf{Cl}(\mathbb{K}_1)$	$\mathbf{Cl}(\mathbb{K}_2)$	$\mathbf{Cl}(\mathbb{K}_3)$	$\mathbf{Cl}(\mathbb{K}_4)$	$\mathbf{Cl}(\mathbb{K}_5)$	$\mathbf{Cl}(\mathbb{K}_6)$	$\mathbf{Cl}(\mathbb{K}_7)$
2.5.269	1	3, 1	(30, 10, 2)	(330, 2, 2)	(120, 12)	(30, 2, 2)	(30, 2, 2)	(30, 2, 2)	(30, 2, 2)
2.53.29	1	3, 1	(78, 2, 2)	(78, 6, 2)	(104, 4)	(130, 2, 2)	(26, 2, 2)	(26, 2, 2)	(130, 2, 2)
2.5.389	1	3, 1	(42, 14, 2)	(462, 2, 2)	(168, 4)	(42, 2, 2)	(42, 6, 2)	(42, 6, 2)	(42, 2, 2)
2.53.37	1	4, 1	(30, 10, 2)	(30, 2, 2)	(80, 4)	(10, 2, 2)	(30, 2, 2)	(30, 2, 2)	(10, 2, 2)
2.13.157	1	4, 1	(26, 26, 2)	(78, 6, 2)	(208, 4)	(26, 2, 2)	(26, 2, 2)	(26, 2, 2)	(26, 2, 2)
2.13.269	1	4, 1	(90, 10, 2)	(990, 6, 2)	(720, 4)	(90, 2, 2)	(90, 2, 2)	(90, 2, 2)	(90, 2, 2)

Table 8: Invariants of \mathbb{K}_j for the case $(\frac{p_1}{p_2}) = 1$ and $(\frac{\pi_1}{\pi_3}) = 1$

$d = 2 \cdot p_1 \cdot p_2$	q	m, n	$\mathbf{Cl}(\mathbb{K}_1)$	$\mathbf{Cl}(\mathbb{K}_2)$	$\mathbf{Cl}(\mathbb{K}_3)$	$\mathbf{Cl}(\mathbb{K}_4)$	$\mathbf{Cl}(\mathbb{K}_5)$	$\mathbf{Cl}(\mathbb{K}_6)$	$\mathbf{Cl}(\mathbb{K}_7)$
4498 = 2.13.173	1	2, 2	(210, 2, 2)	(42, 14, 2)	(280, 4)	(28, 2)	(28, 2)	(28, 2)	(28, 2)
4610 = 2.461.5	2	2, 4	(390, 2, 2)	(234, 2, 2)	(2496, 4)	(780, 2)	(260, 2)	(260, 2)	(780, 2)
5090 = 2.5.509	2	2, 2	(234, 2, 2)	(390, 2, 2)	(624, 4)	(52, 2)	(156, 2)	(156, 2)	(52, 2)
5410 = 2.541.5	1	2, 3	(110, 2, 2)	(22, 22, 2)	(1584, 4)	(44, 2)	(44, 2)	(44, 2)	(44, 2)
6322 = 2.29.109	1	2, 2	(90, 6, 2)	(18, 6, 2)	(504, 4)	(180, 2)	(36, 6)	(36, 6)	(180, 2)
7090 = 2.709.5	2	2, 2	(130, 2, 2)	(442, 2, 2)	(1872, 4)	(52, 2)	(52, 2)	(52, 2)	(52, 2)

Table 9: Invariants of \mathbb{L}_j for $(\frac{p_1}{p_2}) = 1$ and $(\frac{\pi_1}{\pi_3}) = 1$, m is always 2

d	q	b	n	$Cl(\mathbb{L}_1)$	$Cl(\mathbb{L}_2)$	$Cl(\mathbb{L}_3)$	$Cl(\mathbb{L}_4)$	$Cl(\mathbb{L}_5)$	$Cl(\mathbb{L}_6)$	$Cl(\mathbb{L}_7)$
17090	2	-1	2	(9240, 420)	(4620, 2)	(4620, 2)	(420, 42)	(420, 42)	(1680, 30)	(1680, 10)
1586	1	1	2	(660, 12)	(220, 2)	(132, 6)	(132, 6)	(132, 6)	(88, 2)	(88, 2)
2290	1	-1	2	(780, 60)	(260, 2)	(60, 10)	(60, 10)	(60, 10)	(120, 2)	(120, 2)
2626	2	1	2	(504, 12)	(252, 6)	(252, 6)	(36, 6)	(36, 6)	(144, 6)	(144, 6)
4610	2	1	4	(18720, 12)	(2340, 30)	(2340, 30)	(780, 30)	(780, 30)	(12480, 30)	(12480, 10)
5410	1	-1	3	(3960, 44)	(44, 22)	(44, 22)	(220, 2)	(220, 2)	(1584, 2)	(1584, 2)

Table 10: Invariants of \mathbb{L}_j for $(\frac{p_1}{p_2}) = 1$ and $(\frac{\pi_1}{\pi_3}) = -1$

d	b	m, n	$Cl(\mathbb{L}_1)$	$Cl(\mathbb{L}_2)$	$Cl(\mathbb{L}_3)$	$Cl(\mathbb{L}_4)$	$Cl(\mathbb{L}_5)$	$Cl(\mathbb{L}_6)$	$Cl(\mathbb{L}_7)$
1090	1	4, 1	(240, 6)	(30, 6, 2)	(6, 6, 6)	(6, 6, 6)	(6, 6, 6)	(24, 12)	(48, 6)
1490	-1	3, 1	(504, 6)	(126, 6, 2)	(126, 6, 2)	(18, 6, 6)	(18, 6, 6)	(72, 2)	(36, 12)
4082	-1	4, 1	(624, 78)	(26, 26, 2)	(78, 6, 2)	(78, 6, 2)	(78, 6, 2)	(208, 2)	(104, 4)
4706	1	3, 1	(6120, 6)	(306, 6, 2)	(510, 6, 2)	(510, 6, 2)	(510, 6, 2)	(204, 4)	(408, 2)
6994	-1	4, 1	(7920, 30)	(90, 10, 2)	(990, 6, 2)	(990, 6, 2)	(990, 6, 2)	(720, 2)	(360, 4)
7474	-1	5, 1	(43680, 2)	(78, 6, 2)	(2730, 2, 2)	(2730, 2, 2)	(2730, 2, 2)	(1248, 6)	(208, 4)

Table 11: Invariants of \mathbb{K}_j for the case $(\frac{p_1}{p_2}) = -1$

$d = 2.p_1.p_2$	q	π	m	$Cl(\mathbb{K}_1)$	$Cl(\mathbb{K}_2)$	$Cl(\mathbb{K}_3)$	$Cl(\mathbb{K}_4)$	$Cl(\mathbb{K}_5)$	$Cl(\mathbb{K}_6)$	$Cl(\mathbb{K}_7)$
130 = 2.5.13	2	-1	2	(12, 2)	(4, 2)	(8, 4)	(4, 2)	(2, 2, 2)	(2, 2, 2)	(4, 2)
370 = 2.37.5	1	-1	3	(12, 2)	(60, 2)	(24, 4)	(12, 2)	(6, 2, 2)	(6, 2, 2)	(12, 2)
530 = 2.5.53	2	1	2	(84, 2)	(84, 2)	(56, 4)	(14, 2, 2)	(28, 2)	(28, 2)	(14, 2, 2)
1970 = 2.197.5	2	1	2	(260, 2)	(260, 2)	(312, 12)	(26, 2, 2)	(52, 2)	(52, 2)	(26, 2, 2)
2930 = 2.293.5	1	1	3	(252, 2)	(252, 2)	(56, 4)	(42, 2, 2)	(28, 2)	(28, 2)	(42, 2, 2)
3538 = 2.61.29	1	-1	5	(84, 2)	(420, 2)	(224, 4)	(28, 2)	(14, 2, 2)	(14, 2, 2)	(28, 2)
5570 = 2.5.557	1	-1	4	(660, 2)	(180, 6)	(240, 4)	(60, 2)	(30, 2, 2)	(30, 2, 2)	(60, 2)
6130 = 2.5.613	2	-1	2	(1260, 6)	(180, 2)	(72, 36)	(36, 2)	(18, 2, 2)	(18, 2, 2)	(36, 2)

Table 12: Invariants of \mathbb{L}_j for the case $(\frac{p_1}{p_2}) = -1$

$d = 2.p_1.p_2$	q	π	m	$Cl(\mathbb{L}_1)$	$Cl(\mathbb{L}_2)$	$Cl(\mathbb{L}_3)$	$Cl(\mathbb{L}_4)$	$Cl(\mathbb{L}_5)$	$Cl(\mathbb{L}_6)$	$Cl(\mathbb{L}_7)$
130 = 2.5.13	2	-1	2	(24, 2)	(12, 2)	(12, 2)	(4, 2)	(4, 2)	(8, 2)	(4, 4)
370 = 2.5.37	1	-1	3	(120, 2)	(60, 2)	(12, 2)	(12, 2)	(12, 2)	(24, 2)	(12, 4)
530 = 2.5.53	2	1	2	(168, 6)	(84, 2)	(84, 2)	(84, 2)	(84, 2)	(28, 4)	(56, 2)
1970 = 2.5.197	2	1	2	(1560, 30)	(260, 2)	(260, 2)	(260, 2)	(260, 2)	(156, 12)	(312, 6)
2930 = 2.293.5	1	1	3	(504, 18)	(252, 6)	(252, 6)	(252, 6)	(252, 6)	(84, 12)	(56, 2)
3538 = 2.29.61	1	-1	5	(3360, 6)	(420, 2)	(420, 2)	(84, 2)	(84, 2)	(224, 2)	(112, 4)
5570 = 2.557.5	1	-1	4	(7920, 6)	(180, 6)	(660, 2)	(660, 2)	(660, 2)	(240, 2)	(120, 4)
6130 = 2.5.613	2	-1	2	(2520, 90)	(1260, 6)	(1260, 6)	(180, 2)	(180, 2)	(72, 18)	(36, 36)

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